

# Some remarks on the definition of classical energy and its conservation laws

Mayeul Arminjon

Laboratory “Soils, Solids, Structures, Risks”, 3SR  
(Grenoble-Alpes University & CNRS), Grenoble, France.

**Abstract.** In classical non-relativistic theories, there is an exact local conservation equation for the energy, having the form of the continuity equation for mass conservation, and this equation occurs from the power equation. We illustrate this by the example of Newtonian gravity for self-gravitating elastic bodies. In classical special-relativistic theories, there is also an exact local conservation equation for the energy, though it comes from the definition of the energy-momentum tensor. We then study that definition in a general spacetime: Hilbert’s variational definition is briefly reviewed, with emphasis on the boundary conditions. We recall the difference between the local equation verified by Hilbert’s tensor  $\mathbf{T}$  in a curved spacetime and the true local conservation equations discussed before. We check by a direct calculation that the addition of a total divergence does not change  $\mathbf{T}$  and show a mistake that can be done and that leads to state the contrary. We end with a result proving uniqueness of the energy density and flux, when both depend polynomially on the fields.

## 1. Introduction

In non-relativistic physics, the concept of energy emerges when one considers the power done (the scalar product of the force by the velocity) on a mass point or a volume element. The energy of a mass point appears in the power equation as a natural scalar quantity (the sum  $\frac{1}{2}mv^2 + V$ ), which is conserved in the case of a time-independent potential  $V$ , and is still relevant if the potential depends on time. This is well known. The energy of a continuous medium subjected to internal forces and to an external force field is a volume density and it also emerges from the power done. However, in general, the local conservation of energy then appears in the form of a balance equation, though it is one in which there is no source term. That is, the energy leaving (or entering) a given domain is exactly identified as a flux going through the boundary surface of the domain. We illustrate this point in **Section 2**, first on the example of the energy conservation equation for a self-gravitating elastic medium in Newtonian gravity. In relativistic theories, the volume energy density is essentially the (0 0) component of the energy-momentum-stress tensor, in what follows “the T-tensor” for brevity. In Subsect. 2.3, we show that, in the Minkowski spacetime, the conservation equation verified by the T-tensor is indeed a true local conservation equation of the same kind as in non-relativistic physics.

It is standard to say that the expression of the T-tensor is deduced from a Lagrangian, the latter being assumed to govern the relevant system of matter fields via the principle of stationary action. There are two distinct definitions of a T-tensor from a Lagrangian: (i) The so-called “canonical” or “Noether” tensor, say  $\boldsymbol{\tau}$ , is a by-product of the Euler-Lagrange equations. (ii)

The ‘‘Hilbert tensor’’, say  $\mathbf{T}$ , is the symmetric tensor obtained as the derivative of the matter Lagrangian density with respect to variations of the (spacetime) metric. In **Section 3**, we briefly review the definitions of the canonical and Hilbert tensors from a Lagrangian through the principle of stationary action in a general spacetime. We recall well known facts about the meaning of the standard conservation equation verified by the Hilbert tensor. We argue that one would need local definitions of the energy and momentum densities and their fluxes, in short a local definition of the T-tensor, and one would need also a local conservation equation for the energy. Next, we investigate the consequences of the fact that adding a total divergence to the Lagrangian  $L$  does not change the equations of motion, i.e. the Euler-Lagrange equations. We ask whether one may change  $L$  for another Lagrangian,  $L'$ , so that the associated Hilbert tensor fields  $\mathbf{T}$  and  $\mathbf{T}'$  be different.

Finally, in **Section 4** we investigate if the energy equation is unique for a given system of fields, i.e., if the energy density and fluxes can be considered to be uniquely defined. We show that, if the energy density and its flux depend on the fields (both the matter fields and the ‘‘long-distance’’ fields) in a polynomial way, then they are determined uniquely. We show this by considering separately the contributions of matter (including its potential energy in the long-distance fields) and the long-distance fields.

## 2. Local energy conservation in non-relativistic physics and in special relativity

### 2.1. Local energy balance for an elastic medium or a barotropic fluid in Newtonian gravity

We consider an elastically deformable medium with mass density  $\rho$ , velocity field  $\mathbf{v}$  (with respect to some inertial frame), and Cauchy stress tensor field  $\boldsymbol{\sigma}$ . The motion takes place in a gravitational field, with Newtonian gravity potential  $U$ . Newton’s second law writes then:

$$\rho \frac{d\mathbf{v}}{dt} = \rho \nabla U + \mathbf{div} \boldsymbol{\sigma}, \quad (1)$$

where  $\frac{d}{dt}$  means the ‘‘material’’ derivative. The power (per unit volume) is got by taking the scalar product of (1) with the velocity  $\mathbf{v}$ . The isentropy of the deformation of an elastic medium writes

$$\boldsymbol{\sigma} : \mathbf{grad} \mathbf{v} = \rho \frac{d\Pi}{dt}, \quad (2)$$

where  $\Pi$  is the mass density of elastic energy. Using also the mass conservation:

$$\frac{\partial \rho}{\partial t} + \mathbf{div}(\rho \mathbf{v}) = 0, \quad (3)$$

one may rewrite the expression of the power as

$$\frac{\partial w_m}{\partial t} + \mathbf{div} \boldsymbol{\Phi}_m = -\rho \frac{\partial U}{\partial t}, \quad (4)$$

where

$$w_m := \rho \left( \frac{\mathbf{v}^2}{2} + \Pi - U \right) \quad (5)$$

is the *volume energy density of matter*, including its potential energy in the gravitational field, and

$$\boldsymbol{\Phi}_m := w_m \mathbf{v} - \boldsymbol{\sigma} \cdot \mathbf{v} \quad (6)$$

is (the surface density of) the *matter energy flux*. Thus, we have a balance equation with a source term on the r.h.s. Equation (4) can be found in the literature, see Eq. (66.11) in Fock [1]. Note that a *barotropic perfect fluid*, as is commonly assumed in astrophysics, verifies all equations written above [1] — although it is not truly an elastic medium in the sense that it does not have a reference configuration.

## 2.2. Local energy conservation for a system of elastic media / barotropic fluids in Newtonian gravity

Now we assume that all of the matter that produces the gravitational field is indeed in the form of elastic media or barotropic fluids. (Of course, the characteristics of the media may vary in space.) Thus the point-dependent mass density  $\rho$  is just the source of the gravitational field. It therefore obeys the gravitational field equation, i.e. the Poisson equation:

$$\Delta U = -4\pi G\rho. \quad (7)$$

It follows from (7) that

$$\frac{\partial w_g}{\partial t} + \operatorname{div} \Phi_g = \rho \frac{\partial U}{\partial t}, \quad (8)$$

where

$$w_g := \frac{(\nabla U)^2}{8\pi G} \quad (9)$$

is the *volume energy density of the gravitational field*, and

$$\Phi_g := -\frac{\partial U}{\partial t} \frac{\nabla U}{4\pi G} \quad (10)$$

is the *gravitational energy flux*. Equation (8) may be termed the energy balance equation of the gravitational field. Like (4), this also is a balance equation with a source term. The source term in Eq. (8) is just the opposite of the source term in (4). Therefore, combining (4) with (8), we get the *local energy conservation equation in Newtonian gravity* [2]:

$$\frac{\partial (w_m + w_g)}{\partial t} + \operatorname{div} (\Phi_m + \Phi_g) = 0. \quad (11)$$

There is also a local conservation equation for momentum in Newtonian gravity, see e.g. Refs. [2, 3]. Strangely enough, Eq. (11) does not seem to be written in Refs. [1, 3].

## 2.3. Local conservation equations and the energy-momentum tensor in Minkowski spacetime

Recall that the energy-momentum tensor is a second-order spacetime tensor  $\mathbf{T}$ , preferably symmetric. In the Minkowski spacetime,  $\mathbf{T}$  verifies [4] the *local conservation equation*

$$T^{\mu\nu}_{,\nu} = 0 \quad (\text{in Cartesian coordinates}). \quad (12)$$

This is truly a *local* equation of *conservation*, because in *any* given bounded spatial domain  $\Omega$  (not merely in the whole space, and in fact the integrals below do not necessarily make sense in an unbounded domain), it implies two balance equations without any source term. One is a scalar equation:

$$\frac{d}{dx^0} \left( \int_{\Omega} w \, dV \right) = \int_{\partial\Omega} \Phi \cdot \mathbf{n} \, dS, \quad (13)$$

where

$$w := T^{00}, \quad \Phi := -T^{0i} \partial_i \quad (\text{sum over } i = 1, 2, 3). \quad (14)$$

The other one is a (spatial) vector equation:

$$\frac{d}{dx^0} \left( \int_{\Omega} \mathbf{P} \, dV \right) = \int_{\partial\Omega} \Sigma \cdot \mathbf{n} \, dS, \quad (15)$$

where

$$\mathbf{P} := T^{i0} \partial_i, \quad \Sigma := -T^{ij} \partial_i \otimes \partial_j. \quad (16)$$

Thus in Equations (13) and (15), the change on the l.h.s. is due to the flux through the boundary  $\partial\Omega$  on the r.h.s.. The same equations (13) and (15) apply also to Newtonian gravity. E.g. for (13), it follows immediately from Eq. (11) for the total energy density  $w := w_m + w_g$  and the total energy flux  $\Phi := \Phi_m + \Phi_g$ .

Note that the definitions (14) and (16), as well as the conservation equations (13) and (15), are covariant under general *spatial* coordinate changes. This means that *there is one definition of the energy and momentum (and their fluxes) per reference frame*. It is not specific to special relativity; indeed the energy depends on the reference frame in a general spacetime, be it the classical or the quantum-mechanical energy [5].

### 3. Definition of the energy-momentum tensor from a Lagrangian

#### 3.1. Lagrangian and stationary action principle

Where does the T-tensor come from? Assume the equations of motion for the matter fields  $q^A$  ( $A = 1, \dots, n$ ) derive from a Lagrangian  $L(q^A, \partial_\mu q^A, x^\nu)$  through the *principle of stationary action*:

$$\text{For any variation field } \delta q^A = \delta q^A(x^0, \dots, x^3) \text{ with } \delta q^A|_{\partial U} = 0, \text{ we have } \delta S = 0. \quad (17)$$

Here,  $\partial U$  is the boundary, assumed smooth, of some bounded open set  $U$  in the spacetime (now a general one), and  $S$  is the action:

$$S = S_U := \int_U L \sqrt{-g} \, d^4 x, \quad (18)$$

where  $g := \det(g_{\mu\nu})$ . The stationarity (17) is equivalent (see e.g. [4, 6]) to the Euler-Lagrange equations. In a general spacetime, the latter equations write [7]:

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial q_{,\mu}^A} \right) = \frac{\partial \mathcal{L}}{\partial q^A} \quad (A = 1, \dots, n), \quad \mathcal{L} := L \sqrt{-g}. \quad (19)$$

In Lagrangian theories based on the principle of stationary action, two distinct ‘‘T-tensors’’ may be defined.

#### 3.2. The ‘‘canonical’’ (or ‘‘Noether’’) T-tensor

This object is given by

$$\tau_\mu{}^\nu = q_{,\mu}^A \frac{\partial L}{\partial q_{,\nu}^A} - \delta_\mu^\nu L. \quad (20)$$

When  $-g = 1$  and  $\mathcal{L} = L = L(q^A, \partial_\mu q^A)$  does not depend explicitly on the spacetime position, this object occurs naturally from the derivation of the Euler-Lagrange equations (19), which imply that it verifies the desired local conservation equation  $\tau_{\mu,\nu}{}^\nu = 0$  [4]. However, such an independence happens in practice only in a flat spacetime. Moreover, in fact, this object is not necessarily a tensor — even in a flat spacetime, cf. the case of the electromagnetic field [7]. In a general spacetime,  $\tau$  is a tensor for a scalar field [7] — and also for the Dirac field [8].

#### 3.3. Hilbert’s variational definition of the T-tensor

One imposes a small coordinate change:  $x^\mu \mapsto x^\mu + \delta x^\mu$ , such that  $\delta x^\mu = 0$  at spacetime points  $X$  which do not belong to the bounded open set  $U$ , in which one computes the action (18). Say,  $\delta x^\mu = \epsilon \xi^\mu$  with  $\xi$  any smooth vector field that vanishes if  $X \notin U$ , and  $\epsilon \ll 1$ . (Alternatively,

one may regard the mapping defined in coordinates by  $x^\mu \mapsto x^\mu + \delta x^\mu$  as a diffeomorphism of the spacetime manifold, which coincides with the identity map for  $X \notin U$ .) The open set  $U$  is assumed to have a smooth boundary  $\partial U$ , and both have to be included in the open domain of the coordinate system. Because  $U$  is a regular open set, the assumption “ $\boldsymbol{\xi}(X) = 0$  if  $X \notin U$ ” means exactly, as one may show, that the support of  $\boldsymbol{\xi}$  is included in the compact closure  $\bar{U} = U \cup \partial U$ . Since  $\text{Supp}(\partial_\nu \xi^\mu) \subset \text{Supp}(\xi^\mu)$ , this implies that all derivatives of  $\boldsymbol{\xi}$  also vanish if  $X \notin U$ . It follows that the corresponding change  $\delta g^{\mu\nu}$  in  $g^{\mu\nu}$  {as that change is determined to the first order in  $\epsilon$  by Eq. (94.2) in Ref. [4]} also vanishes if  $X \notin U$ . [Here,  $g^{\mu\nu}$  are the components of the inverse of the metric’s matrix  $(g_{\mu\nu})$ .] Thus, in particular,  $\delta g^{\mu\nu} = 0$  on  $\partial U$ . (Note that this is in general false if one assumes merely that  $\delta x^\mu$  vanishes on  $\partial U$ .) One assumes moreover that the matter fields obey the Euler-Lagrange equations (19), and that the explicit dependence of the matter Lagrangian  $L$  upon the spacetime position is merely through the metric and its first order derivatives, i.e.,  $L = L(q^A, q_{,\mu}^A, g^{\mu\nu}, g^{\mu\nu}_{,\rho})$ . Using the divergence theorem, one then may derive the following expression for the first-order change in the action (18):<sup>1</sup>

$$\delta S_U = \int_U \left[ \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^\rho} \left( \frac{\partial \mathcal{L}}{\partial (g^{\mu\nu}_{,\rho})} \right) \right] \delta g^{\mu\nu} d^4 x, \quad \mathcal{L} := L\sqrt{-g}. \quad (21)$$

This expression [4] leads one to define an energy-momentum tensor  $\mathbf{T}$  (usually called “Hilbert tensor”, though not in Ref. [4]) by [4, 7]:

$$\frac{1}{2}\sqrt{-g}T_{\mu\nu} := \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^\rho} \left( \frac{\partial \mathcal{L}}{\partial (g^{\mu\nu}_{,\rho})} \right). \quad (22)$$

In view of Eq. (21), the r.h.s. of Eq. (22) is often called the variational derivative of  $\mathcal{L}$  and noted  $\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}}$  (e.g. [7], see also [9]). As shown by Eq. (21): For the object  $\mathbf{T}$  whose components  $T_{\mu\nu}$  are defined by Eq. (22), we have for any regular bounded open set  $U$  and for any coordinate change  $\delta x^\mu = \epsilon \xi^\mu$  such that  $\xi^\mu(X)$  vanishes for  $X \notin U$ :

$$\delta S_U = \frac{1}{2} \int_U T_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} d^4 x. \quad (23)$$

If the matter Lagrangian  $L$  is *invariant under general coordinate changes*, then the action  $S$  in Eq. (18) is invariant too, hence the change  $\delta S$  given by Eq. (21) or (23) is zero for any possible coordinate change. Assume, moreover, that the object  $\mathbf{T}$  given by (22) turns out to be indeed a tensor. (Note that this tensorial character is not proved by Landau & Lifshitz [4].) Then, using the expression of  $\delta g^{\mu\nu}$  in terms of the vector field  $\boldsymbol{\xi}$ , and since by assumption the latter vanishes on  $\partial U$ , one gets from (23) [4]:

$$T_{\mu}{}^{\nu}{}_{;\nu} = 0. \quad (24)$$

### 3.4. The impact of covariant derivatives in the conservation equation for $\mathbf{T}$

In contrast with (12) [ $T_{\mu}{}^{\nu}{}_{;\nu} = 0$ , with partial derivatives], Eq. (24) [with covariant derivatives] “does not generally express any conservation law whatever”, as was emphasized by Landau & Lifshitz [4]. Fock [1] used similar words: he noted that the four scalar equations contained in (24) “do not by themselves lead to conservation laws”. To explain it quickly, the presence of

<sup>1</sup> When calculating  $\partial \mathcal{L} / \partial g^{\mu\nu}$ , all sixteen  $g^{\mu\nu}$ ’s ( $0 \leq \mu \leq 3$ ,  $0 \leq \nu \leq 3$ ) are considered as independent variables in  $\mathcal{L}$ , even though  $g^{\mu\nu} = g^{\nu\mu}$ . See Note ‡ on p. 269 in Ref. [4]. Thus, all sixty-four  $g^{\mu\nu}_{,\rho}$ ’s ( $0 \leq \mu \leq 3$ ,  $0 \leq \nu \leq 3$ ,  $0 \leq \rho \leq 3$ ) are also considered as independent variables for the calculation of  $\partial \mathcal{L} / \partial g^{\mu\nu}_{,\rho}$ .

covariant derivatives gives to Eq. (24) the form of (12) *plus source terms*, which are the terms linear in the T-tensor itself (that involve the connection coefficients). Nevertheless, Eq. (24) can be rewritten in the form of (12) after introducing some “pseudo-tensor of the gravitational field”  $\mathbf{t}$ . But the definition of  $\mathbf{t}$  is not unique. And  $\mathbf{t}$  behaves as a tensor only for linear coordinate transformations. As a result, it is generally agreed that Eq. (24) can lead only (under special assumptions, e.g. an asymptotically flat spacetime) to *global* conservation laws, see e.g. [4, 10]. However, in order to be able to investigate the energy balance in any relevant domain, one would need to know uniquely the relevant energy density and its flux. And one would need that they obey a true and local conservation equation. (This is indeed the case in most fields of physics, e.g. in mechanics, thermodynamics, chemistry, electrodynamics, ..., including Newtonian gravitation and special-relativistic physics — as shown in Sect. 2 — and also in some alternative relativistic theories of gravitation.) What is thus lacking in theories based on Eq. (24), which include general relativity and its numerous variants or extensions, is not merely an exact local concept of the *gravitational* energy. As illustrated in Sect. 2, the local concept of energy is associated with a local conservation equation of the type (12) for it, and it is precisely the rewriting of Eq. (24) as such an equation that is neither tensorial nor unique. Note that, if one wants to define the material energy density as the (0 0) component of the tensor  $\mathbf{T}$ , he or she has no way to decide if it should be  $T^{00}$ ,  $T_0^0$ , or  $T_{00}$  — and these do differ numerically, the more so as the gravitational field is stronger. Thus, there is not in these theories an exact local concept for any form of *material energy*, either.

### 3.5. Is there an ambiguity in defining the Hilbert tensor from a Lagrangian?

In addition to the difficulty described in the foregoing subsection, which does not seem solvable in the framework of the said theories, there is a point that needs clarification. In a curved spacetime, the Hilbert tensor field  $\mathbf{T}$  is taken as the *source of the gravitational field* — in general relativity and in many other relativistic theories of gravity. Clearly, that source has to be *locally* defined: it is not the global value (the space integral) of  $\mathbf{T}$  that matters to determine the gravitational field, but indeed the distribution of its local value. However, could not the Hilbert tensor be subject to “relocalizations”, due to the fact that the Lagrangian determining the equations of motion is not unique?

Let us add to the matter Lagrangian  $L$  a total divergence:

$$L \mapsto L' = L + \Delta, \quad \Delta = \operatorname{div} \mathbf{V} = \frac{1}{\sqrt{-g}} \partial_\rho (V^\rho \sqrt{-g}), \quad (25)$$

where

$$\mathbf{V} = \mathbf{V}(g^{\mu\nu}, x^\rho) \quad (26)$$

is a spacetime vector field. Then the Euler-Lagrange equations (19) stay *unchanged*, see e.g. Ref. [6]. Note that, of course, the modified Lagrangian  $L'$  is also an invariant scalar if  $L$  is. But, a priori, shouldn't the T-tensor generally change?

Consider the Hilbert tensor (22) and take for vector  $\mathbf{V}$  the simple form

$$V^\rho = A_{\sigma\tau}^\rho g^{\sigma\tau}. \quad (27)$$

Here  $\mathbf{A}$ , with components  $A_{\tau\sigma}^\rho = A_{\sigma\tau}^\rho = A_{\sigma\tau}^\rho(x^\nu)$ , is a (1 2) tensor field, so that of course  $\mathbf{V}$  is indeed a spacetime vector. Changing  $L$  for  $L' = L + \Delta$ , where  $\Delta = \operatorname{div} \mathbf{V}$  and with the vector  $\mathbf{V}$  in Eq. (27), gives by (22) a Hilbert tensor  $\mathbf{T}' = \mathbf{T} + \mathbf{T}''$ , with  $T''_{\mu\nu}$  the Hilbert tensor associated by (22) with the Lagrangian  $\Delta = \operatorname{div} \mathbf{V}$ . Using the well known formula [4]

$$\frac{\partial g}{\partial g^{\mu\nu}} = -g g_{\mu\nu}, \quad \frac{\partial g}{\partial x^\rho} = -g g_{\mu\nu} g_{,\rho}^{\mu\nu}, \quad (28)$$

we get:

$$\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{\sqrt{-g}}{2}g_{\mu\nu}, \quad \frac{\partial\sqrt{-g}}{\partial x^\rho} = -\frac{\sqrt{-g}}{2}g_{\mu\nu}g_{,\rho}^{\mu\nu}. \quad (29)$$

It follows that

$$\Delta := \operatorname{div} \mathbf{V} = A_{\sigma\tau}^\rho \left( g_{,\rho}^{\sigma\tau} - \frac{1}{2}g_{\mu\nu}g_{,\rho}^{\mu\nu}g^{\sigma\tau} \right) + A_{\sigma\tau,\rho}^\rho g^{\sigma\tau}. \quad (30)$$

To obtain the corresponding Hilbert tensor  $T''_{\mu\nu}$  by (22), we have to calculate the derivatives  $\frac{\partial(\sqrt{-g}\Delta)}{\partial g^{\mu\nu}}$  and  $\frac{\partial(\sqrt{-g}\Delta)}{\partial g_{,\rho}^{\mu\nu}}$ . From the expression (30), and using again (29), this looks easy (though somewhat tedious): if we begin with calculating just  $\frac{\partial}{\partial g^{\mu\nu}}(g_{\sigma\tau}g_{,\rho}^{\sigma\tau})$ , we note that

$$g_{\sigma\tau}g_{,\rho}^{\sigma\tau} = -g^{\sigma\tau}g_{\sigma\tau,\rho}, \quad (31)$$

and it seems that we have simply

$$\frac{\partial}{\partial g^{\mu\nu}}(g^{\sigma\tau}g_{\sigma\tau,\rho}) = g_{\mu\nu,\rho}. \quad (32)$$

Using this ‘‘obvious’’ relation, it had been found in the first version of this work that  $T''_{\mu\nu} \neq 0$ , i.e., that the addition of a total divergence would change the Hilbert tensor (moreover, in a non-tensorial way). However, the partial derivative on the l.h.s. of (32) is that of a piece of Lagrangian, considered as a function of the variables  $g^{\sigma\tau}$  and  $g_{,\rho}^{\sigma\tau}$ . Thus, that partial derivative  $\frac{\partial}{\partial g^{\mu\nu}}$  is defined by keeping constant all components  $g^{\sigma\tau}$  such that  $(\sigma, \tau) \neq (\mu, \nu)$ , as well as all derivatives  $g_{,\rho}^{\sigma\tau}$  ( $\sigma, \tau, \rho \in \{0, \dots, 3\}$ ), and by varying just  $s := g^{\mu\nu}$ . But in these conditions, the components of the inverse matrix  $(g_{\sigma\tau}) = (g^{\phi\psi})^{-1}$  vary. Hence, the partial derivatives

$$g_{\sigma\tau,\rho} = -g_{\sigma\phi}g_{,\rho}^{\phi\psi}g_{\psi\tau} \quad (33)$$

also vary. Therefore, Eq. (32) is false.

In fact, for a Lagrangian which is the total divergence of a vector field having the general form (26), one can show by direct calculation that the Hilbert tensor is nil. To do this properly, one should account explicitly for the meaning of Eq. (26). Namely, one assumes given some function  $\underline{\mathbf{V}} = \underline{\mathbf{V}}(\mathbf{X}, \underline{\mathbf{g}})$  of the arguments  $\mathbf{X} = (x^\mu) \in \mathbb{R}^4$  and  $\underline{\mathbf{g}} := (g^{\mu\nu}) \in \mathbb{R}^{16}$ . For each given metric field, a vector field  $\mathbf{V}$  is got by substituting for  $\underline{\mathbf{g}}$  the inverse matrix  $\mathbf{g}(\mathbf{X}) := (g^{\mu\nu}(\mathbf{X}))$  of the metric tensor at the spacetime coordinate point  $\mathbf{X}$ , thus  $\mathbf{V}(\mathbf{X}) = \underline{\mathbf{V}}(\mathbf{X}, \mathbf{g}(\mathbf{X}))$ . The total divergence of  $\underline{\mathbf{V}}$  is a differential function [11] defined through the total derivatives of  $\underline{\mathbf{V}}$ :

$$\underline{\Delta} := \frac{1}{\sqrt{-\underline{g}}}D_\rho(\sqrt{-\underline{g}}\underline{V}^\rho) = \frac{1}{\sqrt{-\underline{g}}}D_\rho\underline{U}^\rho, \quad \underline{g} := \det \underline{\mathbf{g}}^{-1}, \quad \underline{U}^\rho := \sqrt{-\underline{g}}\underline{V}^\rho = \underline{U}^\rho(\mathbf{X}, \underline{\mathbf{g}}), \quad (34)$$

with

$$D_\rho\underline{U}^\rho = \partial_\rho\underline{U}^\rho + \frac{\partial\underline{U}^\rho}{\partial g^{\sigma\tau}}\underline{g}_\rho^{\sigma\tau}. \quad (35)$$

[Here the  $\underline{g}_\rho^{\sigma\tau}$ ’s are the values that may be taken by the partial derivatives  $g_{,\rho}^{\sigma\tau}$  once one substitutes  $g^{\mu\nu}(\mathbf{X})$  for  $\underline{g}^{\mu\nu}$ . The  $\underline{g}_\rho^{\sigma\tau}$ ’s, or the matrices  $\underline{\mathbf{g}}_\rho := (g_{,\rho}^{\sigma\tau})$ , are arguments of  $\underline{\Delta}$ .] Thus, the Hilbert tensor associated with the Lagrangian  $\underline{\Delta}$  can also be seen as a differential function, say  $\underline{\mathbf{T}}''$ . The tensor field  $\mathbf{T}'' = \mathbf{T}''(\mathbf{X})$  is then got by substituting the function  $\mathbf{g} = \mathbf{g}(\mathbf{X})$  for  $\underline{\mathbf{g}}$ , and the matrices  $\mathbf{g}_{,\rho}(\mathbf{X})$  for the matrices  $\underline{\mathbf{g}}_\rho$ . From (22), the components of  $\underline{\mathbf{T}}''$  are given by

$$\underline{T}''_{\mu\nu} = \frac{2}{\sqrt{-\underline{g}}} \left( \frac{\partial \mathcal{D}}{\partial \underline{g}^{\mu\nu}} - D_\rho \left( \frac{\partial \mathcal{D}}{\partial \underline{g}_{,\rho}^{\mu\nu}} \right) \right), \quad \mathcal{D} := D_\rho \underline{U}^\rho = \mathcal{D}(\mathbf{X}, \underline{\mathbf{g}}, (\underline{\mathbf{g}}_\rho)). \quad (36)$$

From (35) and (36)<sub>2</sub>, we get

$$\frac{\partial \mathcal{D}}{\partial \underline{g}^{\mu\nu}} = \frac{\partial}{\partial \underline{g}^{\mu\nu}} (D_\rho \underline{U}^\rho) = \frac{\partial}{\partial \underline{g}^{\mu\nu}} (\partial_\rho \underline{U}^\rho) + \frac{\partial^2 \underline{U}^\rho}{\partial \underline{g}^{\mu\nu} \partial \underline{g}^{\sigma\tau}} \underline{g}_\rho^{\sigma\tau} = D_\rho \left( \frac{\partial \underline{U}^\rho}{\partial \underline{g}^{\mu\nu}} \right). \quad (37)$$

Also from (35) and (36)<sub>2</sub>, we still get:

$$\frac{\partial \mathcal{D}}{\partial \underline{g}_\rho^{\mu\nu}} = \frac{\partial}{\partial \underline{g}_\rho^{\mu\nu}} (D_\sigma \underline{U}^\sigma) = \frac{\partial}{\partial \underline{g}_\rho^{\mu\nu}} \left( \partial_\sigma \underline{U}^\sigma + \frac{\partial \underline{U}^\sigma}{\partial \underline{g}_\rho^{\tau\phi}} \underline{g}_\sigma^{\tau\phi} \right) = \frac{\partial \underline{U}^\rho}{\partial \underline{g}_\rho^{\mu\nu}}. \quad (38)$$

From (36)<sub>1</sub>, (37) and (38), it follows that  $\underline{T}''_{\mu\nu} = 0$ , hence also  $T''_{\mu\nu} = 0$ , as announced. In a forthcoming paper, we will give a ‘‘variational’’ proof of this same result.

#### 4. A uniqueness result for the energy balance

##### 4.1. Is the energy balance equation unique?

We begin with a discussion of this question for a system of elastic media and barotropic fluids in Newtonian gravity (NG). The energy balance (4) established in Section 2 for the *matter field equations* of NG has the form

$$\partial_\mu V^\mu = \text{field source} := -\rho \frac{\partial U}{\partial t}, \quad (39)$$

with the four-components column vector ( $V^\mu$ ) being here the ‘‘matter current’’ made with the matter energy density and flux: ( $V^\mu$ ) = ( $w_m, \Phi_m$ ). As we have indicated, Eq. (39) [i.e. Eq. (4)] is verified as soon as the following three equations are verified among the matter field equations: Newton’s second law (1), the isentropy equation (2), and the continuity equation (3). We note that, in view of Eqs. (5) and (6), the matter current ( $V^\mu$ ) is polynomial in the fields  $q^A$  ( $A = 1, \dots, n$ ) = ( $\rho, \mathbf{v}, \Pi, \boldsymbol{\sigma}, U$ ) that appear in those equations. (Thus  $n = 12$  here. The gravitational potential  $U$  plays the same role as does the metric tensor in a Lagrangian for the matter fields in a curved spacetime, as was the case in the foregoing section.) Now we ask: Can we change the matter current  $V^\mu$  for another one  $V'^\mu = V^\mu + W^\mu$ , also polynomial with respect to the  $q^A$ ’s, in such a way that the l.h.s. of (39) would be unchanged *for whatever values  $q^A$  of the fields?* I.e., can we find a column four-vector  $W^\mu$  so that we have  $\partial_\mu W^\mu \equiv 0$ ?

##### 4.2. A uniqueness result

Thus, let  $W^\mu$  be an order- $N$  polynomial in the field values at the spacetime point  $X$  that is being considered:

$$W^\mu(X, q^A) = C_0^\mu + C_{1A}^\mu q^A + \dots + C_{N A_1 \dots A_N}^\mu q^{A_1} \dots q^{A_N} \quad (A_1 \leq \dots \leq A_N), \quad (40)$$

it being understood that  $q^A = q^A(X)$  and that the coefficients  $C_0^\mu, \dots, C_{N A_1 \dots A_N}^\mu$  also may depend on  $X$ . Assume that its 4-divergence vanishes identically,  $\partial_\mu W^\mu \equiv 0$ :

$$0 \equiv C_{0,\mu}^\mu + C_{1A,\mu}^\mu q^A + C_{1A}^\mu q_{,\mu}^A + \dots + C_{N A_1 \dots A_N, \mu}^\mu q^{A_1} \dots q^{A_N} + C_{N A_1 \dots A_N}^\mu (q_{,\mu}^{A_1} q^{A_2} \dots q^{A_N} + \dots + q^{A_1} \dots q^{A_{N-1}} q_{,\mu}^{A_N}). \quad (41)$$

Consider a fixed spacetime point  $X(x^\rho)$ . The polynomial got by substituting  $Y^A = q^A(X)$ ,  $Z_\mu^A = q_{,\mu}^A(X)$  ( $A = 1, \dots, n$ ;  $\mu = 0, \dots, 3$ ) into the r.h.s. of (41) is identically zero. Hence its coefficients are all zero. In particular:

$$C_{1A}^\mu = 0, \dots, C_{N A_1 \dots A_N}^\mu = 0. \quad (42)$$



Thus all coefficients in (40) are zero — except perhaps  $C_0^\mu$ , with  $C_{0,\mu}^\mu = 0$ .

We thus got that we *cannot* alter the analytical expression of  $w_m$  and  $\Phi_m$  on the l.h.s. of the matter energy balance (39). [Apart from arbitrarily adding a zero-divergence vector field  $C_0^\mu$  that is *independent* of the matter fields — this is indeed obviously possible, but we can get rid of this by asking that the matter current ( $V^\mu$ ) be polynomial in the fields *and have no zero-order term*, as is indeed the case in all concrete examples.] The gravitational energy balance (8) has just the same form:

$$\partial_\mu V^\mu = \text{matter source} := \rho \frac{\partial U}{\partial t}, \quad (43)$$

where  $V^\mu = (w_g, \Phi_g)$  is polynomial in the gravitational field  $q^A$  ( $A = 1, \dots, 4$ ) =  $(\partial_\mu U)$ . It is valid when the gravitational field equation is. Therefore, similarly as we found for the matter field energy balance, we cannot alter the analytical expression (8) of the gravitational energy balance.

### 4.3. Generalization

These results are clearly general. Consider e.g. the Maxwell electromagnetic field instead of the Newtonian gravitational field. The energy balance of the e.m. field is:

$$\frac{\partial w_{\text{em}}}{\partial t} + \text{div } \Phi_{\text{em}} = -\mathbf{j} \cdot \mathbf{E}, \quad (44)$$

with  $w_{\text{em}} := \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi}$  the *volume energy density of the electromagnetic field*, and  $\Phi_{\text{em}} := \frac{\mathbf{E} \wedge \mathbf{B}}{4\pi}$  the *electromagnetic energy flux*. The same uniqueness result says that we cannot find an alternative expression for  $w_{\text{em}}$  and  $\Phi_{\text{em}}$  on the l.h.s., which would be valid for whatever values of the fields  $\mathbf{E}$  and  $\mathbf{B}$  and their first derivatives.

## 5. Conclusion

In Section 2, we discussed the conservation of energy in non-relativistic classical physics, taking the rather general example of Newtonian gravity for self-gravitating elastic bodies or barotropic fluids. We then discussed it also in special relativity. In this huge part of physics, we do have local conservation equations for energy: “In any given domain, the loss or gain of energy is due to the flux through the boundary surface of the domain”. In classical special-relativistic theories, this goes through the energy-momentum tensor  $\mathbf{T}$  and its local conservation equation. We noted the important fact that energy, momentum, and their fluxes depend on the reference frame.

In Section 3, we briefly reviewed the definition of the T-tensor from a Lagrangian in theories starting from the principle of stationary action. The basic facts about the canonical tensor were recalled. The definition of the Hilbert tensor  $\mathbf{T}$  was discussed, noting the importance of the boundary condition to be imposed on the coordinate variation field (or infinitesimal diffeomorphism). We recalled the reason why the local equation (24) verified by  $\mathbf{T}$  is not a true conservation equation as is (12), so that an exact local concept of energy (material or gravitational) does not exist in general relativity — unless someone eventually finds some true conservation equation as is (12) [12]. Then we asked if the definition of the Hilbert tensor  $\mathbf{T}$  from a matter Lagrangian  $L$  might be non-unique as is  $L$ , since the latter can be augmented with a four-divergence without altering the Euler-Lagrange equations. We found the (somewhat tricky) mistake in our calculation, in the first version of this work, of the Hilbert tensor for the divergence of a vector field that depends linearly on the inverse of the metric tensor. We also showed by direct calculation that, most generally, the Hilbert tensor (22) is left unchanged by the addition of a four-divergence.

Finally, in Section 4 we began a study on the uniqueness of the definition of the energy density and its flux, and got a first result. Energy local conservation equations often have the form: Matter energy balance + long-distance-field energy balance = 0. If each among the two parts depends polynomially on the relevant fields, then the functional form of the energy density and its flux is unique.

## References

- [1] Fock V I 1959 *The Theory of Space, Time and Gravitation* (First English edition), Pergamon, Oxford. (Russian original edition: Fock V I 1955 *Teoriya Prostranstva, Vremeni i Tyagotenie*, Gos. Izd. Tekhn.-Teoret. Liter., Moskva)
- [2] Arminjon M 1996 Energy and equations of motion in a tentative theory of gravity with a privileged reference frame *Arch. Mech.* **48** 25–52
- [3] Chandrasekhar S 1969 Conservation laws in general relativity and in the post-Newtonian approximations *Astrophys. J.* **158** 45–54
- [4] Lifshitz E M and Landau L D 1971 *The Classical Theory of Fields* (Third English edition), Pergamon, Oxford etc. (Second Russian edition: Landau L D and Lifshits E M 1948 *Teoriya Polya*, Gostekhizdat, Moskva)
- [5] Arminjon M 2015 Some remarks on quantum mechanics in a curved spacetime, especially for a Dirac particle *Int. J. Theor. Phys.* **54** 2218–2235
- [6] Olver P J 2012 *Lectures on Lie Groups and Differential Equations*, Chapter 5: Symmetries of variational problems, Online course
- [7] Leclerc M 2006 Canonical and gravitational stress-energy tensors *Int. J. Mod. Phys. D* **15** 959–990
- [8] Arminjon M 2013 A simpler solution of the non-uniqueness problem of the covariant Dirac theory *Int. J. Geom. Meth. Mod. Phys.* **10**, No. 7, 1350027
- [9] Forger M and Römer H 2004 Currents and the energy-momentum tensor in classical field theory: a fresh look at an old problem *Ann. of Phys.* **309** 306–389
- [10] Stephani H 1982 *General Relativity – An Introduction to the Theory of the Gravitational Field* (English edition), Cambridge University Press, Cambridge. (German original edition: Stephani H 1977 *Allgemeine Relativitätstheorie – Eine Einführung in die Theorie des Gravitationsfeldes*, VEB Deutscher Verlag der Wissenschaften, Berlin)
- [11] Olver P J 2012 *Lectures on Lie Groups and Differential Equations*, Chapter 3: Jets and differential invariants, Online course
- [12] Padmanabhan T 2015 Momentum density of spacetime and the gravitational dynamics *Preprint arXiv:1506.03814v1* (gr-qc)