Reference frames in a general spacetime and the notion of space

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3rd International Conference on Theoretical Physics

"Theoretical Physics and its Applications"

Moscow State Open University, June 24-28, 2013

Motivation and State of the Art

- A reference frame: essentially *a three-dimensional network of observers equipped with clocks and meters.*

- $\exists$ an *associated space* in which the observers are at rest.

- Clearly, both are fundamental notions for physics!

- In Newtonian physics, consideration is usually (not always) restricted to *rigid* reference frames (w.r.t. Euclidean metric).

- In special relativity also: there one considers mainly the *inertial frames*. Each of them is rigid w.r.t. the *spatial metric*.

Motivation and State of the Art (continued)

- Relativistic theories of gravitation: the metric is a field, i.e. it depends on the spacetime position.

- Hence, rigid reference frames are not relevant any more.

- Relevant notion: *reference fluid.* The 3D network is defined by *a time-like vector field* v *on spacetime* (Cattaneo 1958, Massa 1974, Mitskievich 1996): $v =$ unit tangent vector field to the world lines of the points belonging to the network.

- For Landau & Lifshitz (1951) and Møller (1952), a *coordinate system* (or *chart*) defined a reference frame. The link with the definition by a 4-velocity vector field v was done by Cattaneo, 1958:

Motivation and State of the Art (continued)

- Namely, any admissible chart on the spacetime, $\chi : X \mapsto (x^\mu)$ $(\mu = 0, ..., 3)$, defines a unique reference fluid, given by its four-velocity field v: the components of v in the chart χ are

$$v^0 \equiv \frac{1}{\sqrt{g_{00}}}, \qquad v^j = 0 \quad (j = 1, 2, 3). \tag{1}$$

The vector (1) is invariant under the "internal changes"

$$x'^0 = \phi((x^\mu)), \quad x'^k = \phi^k((x^j)) \quad (j, k = 1, 2, 3). \tag{2}$$

- This is valid only within the domain of definition of the chart χ — an open subset U of the whole spacetime manifold V.

Motivation and State of the Art (end)

- Moreover, the notion of the *space associated* with a reference fluid/network was missing in that context. Only a notion of a "spatial tensor" had been defined (Massa 1974, Jantzen-Carini-Bini 1992):

  Namely, a spatial tensor at $X \in \mathrm{V}$ was defined as a *spacetime tensor* which equals its projection onto the hyperplane $\mathrm{H}_X \equiv v(X)^{\perp}$.

- A number of time derivatives along a trajectory can then be introduced. Difficult to choose among them.

Need for a better definition of a reference frame

- Defining a "reference fluid" through its 4-velocity field is correct but unpractical.

- Fixing a "reference frame" by the data of a chart is practical but:

  What is physical here? Is there an associated space? What if we change the chart?

A sketch of the definition of a space manifold

- The 3D space manifold N associated with a reference fluid $\mathcal{F}$ (network of observers) was introduced as the set of the world lines of the points of the network (MA 1996).

- Thus an *element* (point) of N is a *line* of the spacetime manifold V. Spatial tensor fields were defined simply as tensor fields on the spatial manifold N.

- "Sketch" at that time: because the network, hence also N, was defined "physically" and it was not proved that N is indeed a differentiable manifold.

Applications of that notion of a space manifold

- It was noted that the spatial metric defined in Landau & Lifshitz (1951) and in Møller (1952) endows this manifold N with a time-dependent Riemannian metric, thus with a one-parameter family of metrics.

- Then, just one time derivative along a trajectory appears naturally (MA 1996). This allowed us to unambiguously define *Newton's second law* in a general spacetime.

A rigorous and practical definition of a reference frame

- One may define a reference frame as being an *equivalence class of charts* which
  - are all defined on a given open subspace U of the spacetime V;
  - *and* are related 2-by-2 by a *purely spatial* coordinate change:

$$x'^0 = x^0, \quad x'^k = \phi^k((x^j)). \tag{3}$$

- This does define an equivalence relation (MA & F Reifler 2011a)

A definition of a reference frame (continued)

- Thus a reference frame F, i.e. an equivalence class for this relation, can indeed be given by the data of one chart $\chi : X \mapsto (x^\mu)$ with its domain of definition U (an open subset of the spacetime manifold V):

  Namely, F is the equivalence class of (χ, U).
  I.e., F is the set of the charts χ'

  - which are defined on U
  - and such that the transition map $f \equiv \chi' \circ \chi^{-1} \equiv (\phi^\mu)$ corresponds with a purely spatial coordinate change: $x'^0 = x^0, \quad x'^k = \phi^k((x^j))$.

The associated space manifold

- The physics in the former definition: any world line

$$x^j = \mathrm{Constant}\ (j = 1, 2, 3),\ x^0\ \mathrm{variable} \tag{4}$$

is that of an observer.

The corresponding 4-velocity field is given by (1).

This is invariant under the “internal changes” (2).
$\Rightarrow$ a fortiori invariant under the purely spatial coordinate changes (3).

The space manifold $\mathrm{M} = \mathrm{M}_\mathrm{F}$ is the set of the world lines (4).

- In detail: let $P_S : \mathrm{R}^4 \to \mathrm{R}^3, \mathbf{X} \equiv (x^\mu) \mapsto \mathbf{x} \equiv (x^j)$, be the spatial projection.

A world line l is an element of $\mathrm{M_F}$ iff there is a chart $\chi \in \mathrm{F}$ and a triplet $\mathbf{x} \equiv (x^j) \in \mathrm{R}^3$, such that l is the set of *all* points X in the domain U, whose spatial coordinates are $\mathbf{x}$:

$$l = \{\, X \in \mathrm{U};\ P_S(\chi(X)) = \mathbf{x}\,\}. \tag{5}$$

M_F is a differentiable manifold: sketch of the proof

- Let a chart $\chi \in \mathrm{F}$. With any world line $l \in \mathrm{M}_\mathrm{F}$, let us associate the triplet $\mathbf{x} \equiv (x^j)$ made with the *constant* spatial coordinates of the points $X \in l$:

$$\widetilde{\chi} : \mathrm{M}_\mathrm{F} \rightarrow \mathsf{R}^3, \quad l \mapsto \mathbf{x} \text{ such that } \forall X \in l,\ \chi^j(X) = x^j \ (j = 1, 2, 3). \tag{6}$$

- Through Eq. (5), the world line $l \in \mathrm{M}_\mathrm{F}$ is determined uniquely by the data $\mathbf{x}$. I.e., the mapping $\widetilde{\chi}$ is one-to-one.

- One then shows that the set of the mappings $\widetilde{\chi}$ defines a structure of differentiable manifold on M_F:
The spatial part of any chart $\chi \in \mathrm{F}$ defines a chart $\widetilde{\chi}$ on M_F.

Applications of this result

- A Hamiltonian operator of relativistic QM depends *precisely* (MA & F Reifler 2010) on the reference frame F as just defined here. The Hilbert space $\mathcal{H}$ of quantum-mechanical states is the set of the square-integrable functions defined on the associated *space manifold* M_{F} (MA & F Reifler 2011b).

  Prior to this definition, $\mathcal{H}$ depended on the particular spatial coordinate system. This does not seem acceptable.

- The full algebra of spatial tensors can be defined in a simple way: a spatial tensor field is simply a tensor field on the space manifold M_{F} associated with a reference frame F. E.g., the *rotation rate of a spatial triad* (MA 2011).

Questions left open by that result

- These definitions of a reference frame and the associated space manifold apply to a domain U of V, such that at least one regular chart can be defined over the whole of U. Thus these are *local* definitions. Whence the questions:

- Can the definition of a reference fluid by the data of a global four-velocity field v lead to a global notion of space? If yes, what is the link with the former local notions?

The global space manifold N_v associated with a non-vanishing vector field v

- Given a global vector field v on the manifold V and $X \in \mathrm{V}$, let C_X be the solution of

$$\frac{dC}{ds} = v(C(s)), \qquad C(0) = X \tag{7}$$

  that is defined on the *largest possible* open interval I_X containing X. Call the *range* $l_X \equiv C_X(\mathrm{I}_X) \subset \mathrm{V}$ the "maximal integral line at X". If $X' \in l_X$, then $l_{X'} = l_X$.

- We define N_v as the set of the maximal integral lines of v:

$$\mathrm{N}_v \equiv \{l_X;\ X \in \mathrm{V}\}. \tag{8}$$

Local existence of adapted charts

- A chart χ with domain $\mathrm{U} \subset \mathrm{V}$ is said "v–adapted" iff the spatial coordinates remain constant on any integral line l of v — more precisely, remain constant on $l \cap \mathrm{U}$:

$$\exists \mathbf{x} \equiv (x^j) \in \mathsf{R}^3 : \forall X \in l \cap \mathrm{U}, \quad P_S(\chi(X)) = \mathbf{x}. \tag{9}$$

- For any v–adapted chart χ, the mapping

$$\bar{\chi} : l \mapsto \mathbf{x} \text{ such that (9) is verified} \tag{10}$$

is well defined on $\mathrm{D_U} \equiv \{l \in \mathrm{N}_v ;\ l \cap \mathrm{U} \neq \emptyset\}$. Call the v–adapted chart χ "nice" if the mapping $\bar{\chi}$ is one-to-one.

- **Theorem 1.** *Assume the global vector field* v *on* V *is non-vanishing. Then* (under some reasonable technical assumption regarding the flow of the field v), *for any point* $X \in \mathrm{V}$, *there exists a nice* v*–adapted chart* χ *whose domain is an open neighborhood of* X.

Manifold structure of the global set N_v

- Main thing to prove is compatibility of any two charts $\bar{\chi}$, $\bar{\chi}'$ on N_v, associated with two adapted charts χ, χ' on V.

- In the case of the space manifold M_F associated with a local reference frame F, the compatibility of two associated charts $\widetilde{\chi}$ and $\widetilde{\chi}'$ on M_F was $\simeq$ easy to prove: any world line $l \in \mathrm{M}_\mathrm{F}$ is included in the common domain U of any two charts $\chi, \chi' \in \mathrm{F}$. It follows that $\widetilde{\chi}' \circ \widetilde{\chi}^{-1} = (\phi^k)$.

- In contrast, in the present case, two adapted charts χ and χ' have different domains U and U' and we may have

$$\mathrm{U} \cap \mathrm{U}' = \emptyset, \quad l \cap \mathrm{U} \neq \emptyset, \quad l \cap \mathrm{U}' \neq \emptyset, \qquad (11)$$

so the charts χ and χ' don't compose, but $\bar{\chi}$ and $\bar{\chi}'$ do.

Manifold structure of the global set N_v (continued)

- Solution: Consider $\mathbf{x} \in \bar{\chi}(\mathrm{D_U})$, thus $\exists l \in \mathrm{N}_v$ and $\exists X \in l \cap \mathrm{U}$: $\mathbf{x} = \bar{\chi}(l) = P_S(\chi(X))$. Let $\chi(X) = (t, \mathbf{x})$. We use the flow of the vector field v to associate smoothly with any point Y in some neighborhood $\mathrm{W} \subset \mathrm{U}$ of X, a point $g(Y) \in \mathrm{U}'$.

- Then we may write for $\mathbf{y}$ in a neighborhood of $\mathbf{x}$:

$$(\bar{\chi}' \circ \bar{\chi}^{-1})\,(\mathbf{y}) = P_S(\chi'(g(\chi^{-1}(t, \mathbf{y})))), \tag{12}$$

showing the smoothness of $\bar{\chi}' \circ \bar{\chi}^{-1}$.

- It follows that the maps $\bar{\chi}$ deduced from the adapted charts χ build an atlas on N_v, making it a differentiable manifold.

The local manifold M_F is a submanifold of N_v

- Let v be a non-vanishing vector field on V, and let F be a reference frame made of nice v–adapted charts, all defined on the same open set $\mathrm{U} \subset \mathrm{V}$.

- Let $l \in \mathrm{M}_\mathrm{F}$, thus there is some chart $\chi \in \mathrm{F}$ and some $\mathbf{x} \in \mathsf{R}^3$ such that $l = \{\, X \in \mathrm{U};\ P_S(\chi(X)) = \mathbf{x}\,\}$.
Then for any $X \in l$ we have $l' \equiv l_X \in \mathrm{N}_v$ and $l = l' \cap \mathrm{U}$.
The mapping $I : \mathrm{M}_\mathrm{F} \to \mathrm{N}_v,\ l \mapsto l'$ is an immersion.

- We may identify: $\mathrm{M}_\mathrm{F} \simeq I(\mathrm{M}_\mathrm{F}) \subset \mathrm{N}_v$. That is:
The local space associated with a (local) reference frame is part of the global space associated with a (global) reference fluid.

Conclusion

- Defining a reference fluid from its 4-velocity field has been done since a long time but is not very tractable by itself.
- Defining a reference frame as a class of charts exchanging by spatial coordinate change: both practical and correct.
- Associated space with given ref. fluid *or* given ref. frame: needed to define classical trajectories and quantum space of states. A precise notion did not exist. Here defined from the charts adapted to the ref. fluid or frame.
- Adapted charts with a common domain $\rightarrow$ "local" space. This is a part of a global space associated with the ref. fluid.

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