On quantum mechanics in a curved spacetime, especially for a Dirac particle

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Experimental context

Quantum effects in the classical gravitational field are observed on Earth for neutrons (spin $\frac{1}{2}$ particles) & atoms:

- COW effect: gravity-induced phase shift measured by neutron (1975) and atom (1991a) interferometry;
- Sagnac effect: Earth-rotation-induced phase shift measured by neutron (1979) and atom (1991b) interferometry;
- GRANIT effect: Quantization of the energy levels proved by threshold in neutron transmission through a thin horizontal slit (2002).

These are QM effects ("first-quantized") and are the only observed effects of the gravity-quantum coupling!
Motivates work on curved-spacetime QM.

Hamiltonian in a general spacetime V

Given: a wave eqn on V. Choose local chart $\mathrm{U} \to \mathbb{R}^4, X \mapsto (x^\mu)$

Wave eqn gets local expression with wave funct$^{\mathrm{n}}$ $\Psi = \Psi((x^\mu))$.
Assume it contains only first-order derivatives w.r.t. $t \equiv x^0$.

Rewrite wave eqn as: $\mathrm{i}\frac{\partial \Psi}{\partial t} = \mathrm{H}\Psi$.

$\Rightarrow$ Hamiltonian operator H depends on coordinate system.

H defined only over the domain of the chart, $\mathrm{U} \subset \mathrm{V}$.
But: Ψ may safely be assumed to vanish outside U !

Changing the chart: $(x^\mu) \hookrightarrow (x'^\mu)$ gives $\mathrm{H}' \neq \mathrm{H}$, poorly related.

Hamiltonian in a general spacetime (2/3)

Yet H essentially unchanged under a *purely spatial* change:

$$x'^j = f^j((x^k))\ (j,k=1,2,3), \qquad \text{and} \qquad x'^0 = x^0. \tag{1}$$

E.g., when Ψ behaves as a scalar, $\Psi'((x'^\mu)) = \Psi((x^\nu))$:

$$(\mathrm{H}'\Psi')((x'^\mu)) = (\mathrm{H}\Psi)((x^\nu)). \tag{2}$$

Call "reference frame" an *equivalence class* F *of charts* defined on the same domain U and exchanging by (1).
(MA–F. Reifler, Int. J. Geom. Meth. Mod. Phys. **8**, 155, 2011)

Hamiltonian in a general spacetime (3/3)

A reference frame F in that sense amounts to the data of:

- the domain U of the spacetime,
- The set of the reference world lines "$\mathbf{x} \equiv (x^j) = \mathrm{Const}, \ x^0$ variable",
- A time coordinate map $X \mapsto x^0(X)$ defined for events $X \in \mathrm{U}$.

So this extends the notion of reference frame of classical mechanics.

H depends precisely on the reference frame in this sense.
(MA–F. Reifler, Braz. J. Phys. 2010)

Hamiltonians with a gauge choice (1/4)

Let the wave eqn depend on some "gauge" field G. E.g.: for the covariant Dirac eqn, $G =$ the tetrad field. Assume:

- (i) For any choice of G, there is ($\forall t$) a Hilbert space H of "states" $\Psi = \Psi(\mathbf{x})$, with scalar product: $(\Psi \mid \Phi)$. Thus for another choice $\widetilde{G}$, we get $\widetilde{\mathsf{H}}$, $(\Xi \widetilde{\mid} \Omega)$.

- (ii) Given any two gauge fields, there is ($\forall t$) a *unitary transformation*

$$\mathcal{U} : \mathsf{H} \to \widetilde{\mathsf{H}}. \qquad \forall t,\ \forall \Psi, \Phi \in \mathsf{H}, \quad (\mathcal{U}\Psi \widetilde{\mid} \mathcal{U}\Phi) = (\Psi \mid \Phi). \tag{3}$$

- (iii) Wave eqn is covariant under any admissible unitary transformation $\mathcal{U}$ (change of gauge field: $G \hookrightarrow \widetilde{G}$).

Hamiltonians with a gauge choice (2/4)

When changing the gauge field: $G \hookrightarrow \widetilde{G}$, we get a new form of the wave eqn. Thus, when rewriting it as the Schrödinger eqn, we get a new Hamiltonian operator: $\mathrm{H} \hookrightarrow \widetilde{\mathrm{H}}$.

However, the covariance of the wave eqn under the corresponding unitary transformation $\mathcal{U}$ means this:

$$\text{If} \quad \Xi \equiv \mathcal{U}\Psi, \ \text{then} \quad \mathrm{i}\frac{\partial \Psi}{\partial t} = \mathrm{H}\Psi \quad \Leftrightarrow \quad \mathrm{i}\frac{\partial \Xi}{\partial t} = \widetilde{\mathrm{H}}\Xi \,. \tag{4}$$

In turn, (4) is true iff:

$$\underline{\widetilde{\mathrm{H}} = \mathcal{U}\mathrm{H}\mathcal{U}^{-1} - \mathrm{i}\,\mathcal{U}\left[\partial_t \left(\mathcal{U}^{-1}\right)\right]}. \tag{5}$$

Hamiltonians with a gauge choice (3/4)

Eqn (5): $\widetilde{\mathrm{H}} = \mathcal{U}\mathrm{H}\mathcal{U}^{-1} - \mathrm{i}\,\mathcal{U}\left[\partial_t\left(\mathcal{U}^{-1}\right)\right]$ is well known (in a less general context). It means that H, seen as the *generator of the time evolution,* transforms consistently.

But, seen as an *operator acting on the states* $\Psi = \Psi(\mathbf{x})$, H would transform consistently iff, at any time t, all mean values were invariant under the unitary transformation. Thus, $\mathcal{D}$ being the domain of H, iff:

$$\forall \Psi \in \mathcal{D}, \quad \langle \widetilde{\mathrm{H}} \rangle \equiv (\mathcal{U}\Psi \,\widetilde{|}\, \widetilde{\mathrm{H}}\,(\mathcal{U}\Psi)) = (\Psi \mid \mathrm{H}\,\Psi) \equiv \langle \mathrm{H} \rangle. \qquad (6)$$

Whether H is the Hamiltonian or not, it is not hard to prove that (6) is true iff

$$\widetilde{\mathrm{H}} = \mathcal{U}\mathrm{H}\mathcal{U}^{-1}. \qquad (7)$$

(MA, Int. J. Theor. Phys. **52**, 4032 (2013).)

Hamiltonians with a gauge choice (4/4)

Comparing (5) and (7), we find that H and $\widetilde{\mathrm{H}}$ are equivalent as operators iff

$$\partial_t \mathcal{U} = 0. \tag{8}$$

Thus: *for any wave equation which transforms covariantly under a unitary gauge transformation* $\mathcal{U}$ *depending on the time coordinate, the Hamiltonian operator* H *and its mean values depend on the gauge choice.*

Ambiguity of the Energy Operator

Energy operator $\mathrm{E} \equiv \mathrm{Hermitian\ part\ of\ H}$.
Gauge dependence of $\mathrm{H} \Rightarrow$ gauge dependence of E.

This is not a small effect: for Dirac eqn, difference in the mean values of E for different choices of the tetrad field can be made *arbitrarily large.* (MA, Int. J. Theor. Phys. **52**, 4032 (2013).)

Also true *with electromagnetic field.* (MA, IJTP **53**, 1993 (2014).)
All of this already true in a *Cartesian* chart in a *Minkowski* spacetime.

It means in particular this (MA, IJTP **53**, 1993 (2014)):
The covariant Dirac eqn can't predict the energy levels of the hydrogen atom.

Energy operator vs classical energy (1/2)

The eigenvalues of E are the observable values of the energy of the quantum-mechanical system at hand.

They are associated with stationary solutions of the Schrödinger eqn (when $\mathrm{H} = \mathrm{E}$ is Hermitian).

QM has a strong relation to *classical Hamiltonian mechanics.* E.g. elementary Schrödinger eqn got from the non-relativistic classical Hamiltonian of a test particle:

$$e = H(\mathbf{p}, \mathbf{x}, t) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}, t) \equiv T + V, \qquad \mathbf{p} \equiv m\mathbf{v}, \ldots \tag{9}$$

Energy operator vs classical energy (2/2)

... by applying the classical-quantum correspondence

$$e \to +\mathrm{i}\hbar \frac{\partial}{\partial t}, \qquad p_j \to -\mathrm{i}\hbar \frac{\partial}{\partial x^j}. \tag{10}$$

The correspondence $(10)_1$ leads also directly to interpret the Hamiltonian operator H in the general Schrödinger eqn as the *energy operator* (when $\mathrm{H} = \mathrm{E}$ is Hermitian).

<u>The meaning of the energy operator
is hence inherited from classical Hamiltonian mechanics.</u>

Energy of a classical Hamiltonian particle

In Newtonian physics & in special relativity, energy is conserved:
There is a local energy conservation eqn $\frac{\partial w}{\partial t} + \mathrm{div}\,\mathbf{\Phi} = 0$.

A small piece of matter may be modeled as a *test particle*.
Its energy $e = T + V$: just a part of the whole conserved energy.
However, e is *not* conserved unless V is time-independent.

Also, e depends exactly on the reference frame: e.g., changing the inertial frame changes T, but does not change V.

In a curved ST with e.m. potential V_μ, $e \equiv c\breve{p}_0$ is a Hamiltonian,
$\breve{p}^\mu \equiv mc\frac{\mathrm{d}x^\mu}{\mathrm{d}s} + \frac{q}{c}V^\mu$ (MA–F. Reifler, Braz. J. Phys. **43**, 64 (2013)).
Thus again e depends exactly on the reference frame.

Conclusion

The energy operator E is the quantum equivalent of the Hamiltonian energy e of a classical test particle.

As does e, it depends precisely on the reference frame, and it is in general not constant.

However, in the presence of a gauge choice associated with time-dependent unitary transformations, E is not well defined in a given reference frame.

This indicates that the gauge freedom must be reduced to avoid this case. Proposal in this direction for the Dirac eqn: MA, Int. J. Geom. Meth. Mod. Phys. **10**, 1350027 (2013).