Quantum wave equations in curved space-time from wave mechanics

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Quantum Mechanics (QM) in a gravitational field: ?

- Quantum effects in the classical gravitational field are observed, e.g. on neutrons (interferometry & energy levels)
- Gravity is described by theories with curved space-time.
- Usual way to write QM wave eqs in curved S-T: *covariantization* (connected with *equivalence principle*): searched eqn in curved S-T should coincide with flat-S-T version in coordinates where the connection cancels at X.
- For the Dirac eqn with standard (spinor) transformation, this leads to the Dirac-Fock-Weyl (DFW) eqn, which does *not* obey the equivalence principle.
- Alternatively, in this work we tried to *apply directly the classical-quantum correspondence.*

Dispersion equation of a wave equation

Consider a linear (wave) equation (e.g., of 2nd order):

$$\mathrm{P}\psi \equiv\ a_0(X)\psi + a_1^{\mu}(X)\partial_\mu\psi + a_2^{\mu\nu}(X)\partial_\mu\partial_\nu\psi = 0 \quad (\mu,\nu = 0,...,N), \tag{1}$$

where $X = (t, \mathbf{x}) =$ position in (configuration-)space-time.

Looking for "locally plane-wave" solutions:
$\psi(X) = A\ \exp[i\theta(X)]$, with, at X_0, $\partial_\nu K_\mu(X_0) = 0$, where
$K_\mu \equiv \partial_\mu\theta$: $\mathbf{K} = (K_\mu) = (-\omega, \mathbf{k}) =$ wave (co)vector,

leads to the *dispersion equation:*

$$\Pi_X(\mathbf{K}) \equiv a_0(X) + i\, a_1^{\mu}(X)K_\mu + i^2 a_2^{\mu\nu}(X)K_\mu K_\nu = 0. \tag{2}$$

Substituting $K_\mu \longrightarrow \partial_\mu/i$ determines the linear operator P *uniquely* from the polynomial function $(X, \mathbf{K}) \mapsto \Pi_X(\mathbf{K})$.

The classical-quantum correspondence

The *dispersion relation(s)*: $\omega = W(\mathbf{k}; X)$, fix the wave mode. Obtained by solving $\Pi_X(\mathbf{K}) = 0$ for $\omega \equiv -K_0$. Witham: propagation of $\mathbf{k}$ obeys a *Hamiltonian system:*

$$\frac{\mathrm{d}k_j}{\mathrm{d}t} = -\frac{\partial W}{\partial x^j}, \qquad \frac{\mathrm{d}x^j}{\mathrm{d}t} = \frac{\partial W}{\partial k_j} \qquad (j = 1, ..., N). \tag{3}$$

Wave mechanics: a classical Hamiltonian H describes the skeleton of a wave pattern. Then, the wave eqn should give a dispersion W with the same Hamiltonian trajectories as H. Simplest way to do that: assume that H and W are proportional, $H = \hbar W$... *Leads first to* $E = \hbar\omega$, $\mathbf{p} = \hbar\mathbf{k}$, *i.e.*

$$p_\mu = \hbar K_\mu \qquad (\mu = 0, ..., N). \tag{4}$$

Then, substituting $K_\mu \longrightarrow \partial_\mu / i$, *it leads to the correspondence between a classical Hamiltonian and a wave operator.*

A variant derivation of the Dirac equation

Energy equation of a relativistic particle (*also in a curved space-time*):

$$g^{\mu\nu} p_\mu p_\nu - m^2 = 0 \qquad (c = 1). \tag{5}$$

Dispersion equation associated with this by wave mechanics:

$$g^{\mu\nu} K_\mu K_\nu - m^2 = 0 \qquad (\hbar = c = 1). \tag{6}$$

Applying directly the correspondence $K_\mu \longrightarrow \partial_\mu / i$ to it, leads to the Klein-Gordon eqn. Instead, one may try a *factorization:*

$$\Pi(\mathbf{K}) \equiv (g^{\mu\nu} K_\mu K_\nu - m^2)\mathbf{1} = (\alpha + i\gamma^\mu K_\mu)(\beta + i\zeta^\nu K_\nu). \tag{7}$$

Identifying coeffs. in (7) (with noncommutative algebra), and substituting $K_\mu \longrightarrow \partial_\mu / i$*, leads to the Dirac equation:*

$$(i\gamma^\mu \, \partial_\mu - m)\psi = 0, \qquad \text{with } \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,\mathbf{1}. \tag{8}$$

The classical-quantum correspondence needs using preferred classes of coordinate systems

The dispersion polynomial $\Pi_X(\mathbf{K})$ and the condition $\partial_\nu K_\mu(X) = 0$ stay invariant only inside any class of "infinitesimally-linear" coordinate systems, connected by changes satisfying, at the point $X((x_0^\mu)) = X((x_0'^\rho))$ considered,

$$\frac{\partial^2 x'^\rho}{\partial x^\mu \partial x^\nu} = 0, \qquad \mu, \nu, \rho \in \{0, ..., N\}. \tag{9}$$

One class: *locally-geodesic coordinate systems* at X for $\boldsymbol{g}$, *i.e.*,

$$g_{\mu\nu,\rho}(X) = 0, \qquad \mu, \nu, \rho \in \{0, ..., N\}. \tag{10}$$

Another class occurs if there is a *(physically) preferred reference frame*: that made of changes which are *internal* to this frame.

Conclusion: two new gravitational Dirac equations

I) Assume the first class (locally-geodesic systems). The "rough" Dirac eqn (8), *derived from wave mechanics* in any system of that class, rewrites in a *general* coordinate system as:

$$(i\gamma^\nu D_\nu - m)\,\psi = 0, \qquad (D_\nu\psi)^\mu \equiv \psi^\mu_{;\,\nu} \equiv \partial_\nu\psi^\mu + \Gamma^\mu_{\sigma\nu}\psi^\sigma. \quad (11)$$

($\Gamma^\mu_{\nu\sigma}$'s = Christoffel symbols of $\boldsymbol{g}$.) This eqn *does obey the equivalence principle, in contrast to the standard gravitational extension of the Dirac equation (Dirac-Fock-Weyl eqn).*

II) With the second class (preferred-frame systems), another (preferred-frame) eqn is got. These two eqs are *definitely non-equivalent to the Dirac-Fock-Weyl eqn* – the latter being obtained by the *"covariantization"* of the Dirac eq. with spinor transform. Which is the more correct one? *Experiment* will tell.

Preferred-frame, but generally-covariant, gravitational Dirac equation:

$$(i\gamma^\nu \Delta_\nu - m)\,\psi = 0, \tag{12}$$

where Δ is the unique connection that, *in coordinate systems bound to the preferred reference frame,* is given by

$$\Delta^\mu_{\rho\nu} \equiv \begin{cases} 0 & \text{if } \mu = 0 \text{ or } \nu = 0 \text{ or } \rho = 0 \\ \\ \Delta^j_{lk} & \text{if } \mu = j \text{ and } \nu = k \quad \text{ and } \rho = l \in \{1,2,3\}, \end{cases} \tag{13}$$

the Δ^j_{lk}'s ($j, k, l \in \{1,2,3\}$) being the Christoffel symbols of the *spatial* metric $\boldsymbol{h}$ of the preferred reference frame.