Quantum mechanics for three Dirac equations in a curved spacetime

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Quantum Mechanics (QM) in a gravitational field: ?

- Quantum effects in the classical gravitational field are observed, e.g. neutrons (interferometry, energy levels)
- Gravity is described by theories with curved spacetime (ST).
- Usual way to write QM wave eqs in a curved ST: *covariantization* (connected with *equivalence principle*): searched eqn in curved ST should coincide with flat-ST version in coordinates where the connection cancels at X.
- For the Dirac eqn with standard (spinor) transformation, this leads to the standard eqn (Dirac-Fock-Weyl or DFW), which does *not* obey the genuine equivalence principle.
- In a previous work I got two alternative eqs by *applying directly the classical-quantum correspondence.* Thus 3 eqs.

3 Dirac equations in a curved spacetime

The three gravitational Dirac equations have the same *form:*

$$\gamma^\mu D_\mu \psi = -im\psi, \tag{1}$$

where $\gamma^\mu = \gamma^\mu(X)$ ($\mu = 0, ..., 3$) = field of 4×4 complex matrices defined on the spacetime V, such that

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,\mathbf{1}_4, \quad \mu, \nu \in \{0, ..., 3\} \quad (\mathbf{1}_4 \equiv \mathrm{diag}(1,1,1,1)); \tag{2}$$

and where ψ is a *bispinor* field for the standard, DFW equation but is a **four-vector** field for the two alternative eqs, based on the tensor representation of the Dirac field (TRD);

and D_μ is a covariant derivative, associated with a specific *connection:* one for each of the three equations.

Dirac equation with vector wave function

For Dirac's original equation, the wave function is a (bi)*spinor*.
Due to Dirac matrices γ^μ being assumed Lorentz-invariant.
But a *matrix* should *not* remain invariant after a coord. change!

TRD: the wave function ψ is a (4-)**vector**, and the set of the four Dirac matrices γ^μ builds a third-order **tensor**:

$$\psi'^\mu = \frac{\partial x'^\mu}{\partial x^\sigma}\psi^\sigma, \qquad \gamma'^{\mu\rho}_\nu = \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x'^\rho}{\partial x^\tau}\frac{\partial x^\chi}{\partial x'^\nu}\gamma^{\sigma\tau}_\chi. \qquad (3)$$

This too leaves Dirac equation covariant! MA,Found.Phys.Lett.2006
Moreover, QM predictions in flat ST are **unchanged**, for:
1) *Explicit Dirac eqn is the same,* and 2) *No influence of the possible set of Dirac matrices.* MA & F. Reifler, Braz. J. Phys. 2008

The three different connections

For the two alternative eqs (TRD), this is an affine connection:

$$(D_\mu \psi)^\nu \equiv \partial_\mu \psi^\nu + \Delta^\nu_{\rho\mu} \psi^\rho. \qquad (4)$$

For one of the two TRD eqs (TRD-1), this is the Levi-Civita connection. I.e., $\Delta^\nu_{\rho\mu} = \left\{ {\nu \atop \rho\mu} \right\}$, the Christoffel symbols associated with the spacetime metric $g_{\mu\nu}$.
The corresponding gravitational Dirac eq. *obeys the equivalence principle*.

TRD-2: connection Δ is defined from the *spatial* Levi-Civita connection in an assumed *preferred reference frame*.

DFW (standard eqn): Use the "spin connection", which depends on the γ^μ matrices and is generally *complex*.

A common tool: the hermitizing matrices

For TRD, the set (γ^μ) is a tensor, hence can't be fixed. True even for the "flat" matrices $\gamma^{\sharp\alpha}$ if one defines $\gamma^\mu = a^\mu{}_\alpha\, \gamma^{\sharp\alpha}$ with $a^\mu{}_\alpha$ an orthonormal tetrad. We must be able to use *any* possible set (γ^μ) of Dirac matrices. Note:
also for DFW, one should study the influence of the choice $\gamma^{\sharp\alpha}$.

Solution: use the *hermitizing matrix:* this is a 4×4 matrix A such that

$$A^\dagger = A, \qquad (A\gamma^\mu)^\dagger = A\gamma^\mu \quad \mu = 0, ..., 3, \tag{5}$$

where $M^\dagger \equiv M^{*\,T}$ = Hermitian conjugate of matrix M.
For the usual sets $(\gamma^{\sharp\alpha})$ (Dirac's, "chiral", Majorana), $A = \gamma^{\sharp 0}$.

Existence of A (and B: for α^μ matrices) has been proved by us for any set in a general metric. MA & F. Reifler, Braz. J. Phys. 2008

Definition of the "probability" current

In a flat spacetime, the current is unambiguously defined as

$$J^\mu = \psi^\dagger A \gamma^\mu \psi. \tag{6}$$

The definition (6) is generally-covariant, the current being indeed a *four-vector,* for TRD and for DFW as well. Thus it holds true in a curved spacetime V. (Then γ^μ, A depend on $X \in \mathrm{V}$.)

The current (6) is *independent of the choice of the Dirac matrices:* if one changes one set (γ^μ) for another one $(\tilde{\gamma}^\mu)$, the second set can be obtained from the first one by a point-dependent *similarity transformation:*

$$\exists S = S(X) \in \mathsf{GL}(4, \mathsf{C}) : \qquad \tilde{\gamma}^\mu(X) = S\gamma^\mu(X)S^{-1}, \quad \mu = 0, ..., 3, \tag{7}$$

With the change $\tilde{\psi} = S\psi$, this leaves the current unchanged.

Condition for current conservation

Theorem 1 (MA & FR, arXiv:0807.0570, gr-qc). *Consider the general Dirac equation in a curved spacetime (1), thus either DFW or any of the two TRD equations. In order that any* ψ *solution of (1) satisfy the current conservation*

$$D_\mu J^\mu = 0, \tag{8}$$

it is necessary and sufficient that

$$D_\mu (A\gamma^\mu) = 0. \tag{9}$$

Corollary 1. *For DFW theory, the hermitizing matrix field* $A(X)$ *can be imposed to be some definite* constant *matrix* $A^\sharp$. *Then the current conservation applies to any solution of the DFW equation.*

Admissible (γ^μ, A) fields

Theorem 1 means that *not all possible coefficient fields* (γ^μ, A) *of the Dirac equation are physically admissible, but merely the ones which, in addition to the anticommutation relation (2), satisfy the field equation (9) ensuring current conservation. Such systems we call "admissible."*

Example: in a *flat* spacetime, relevant fields γ^μ are ones which are *constant in Cartesian coordinates* (and hence also the field A). Then the condition for current conservation (9) is satisfied.

If one selects the gamma field at random, the condition (9) and the current conservation do not generally apply to the solutions of the Dirac eqn (1) *even in a flat spacetime*—except for DFW.

The Hamiltonian is frame-dependent

Dirac equation (1) in Schrödinger form:

$$i\frac{\partial\psi}{\partial t} = \mathrm{H}\psi, \qquad (t \equiv x^0), \tag{10}$$

$$\mathrm{H} \equiv m\alpha^0 - i\alpha^j D_j - i(D_0 - \partial_0), \tag{11}$$

where

$$\alpha^0 \equiv \gamma^0/g^{00}, \qquad \alpha^j \equiv \gamma^0\gamma^j/g^{00}. \tag{12}$$

In order that the Hamiltonians H and H' before and after a coordinate change be equivalent operators, the coordinate change must be a **spatial** change: $x'^0 = x^0,\ x'^j = f^j((x^k))$. Then, both sides of the Schrödinger equation (10) behave as a scalar for DFW, and as a vector for TRD. Thus **H depends on the reference frame**, i.e., on the three-dimensional congruence of world lines (observers) which is considered. A general result.

Hermiticity condition for the Hamiltonian

Theorem 5. *A* necessary *condition for the scalar product of time-independent wave functions to be time independent and for the Hamiltonian* H *to be a Hermitian operator, is that the scalar product should be*

$$(\psi \mid \varphi) \equiv \int_{\mathsf{R}^3} \psi^\dagger A \gamma^0 \varphi \ \sqrt{-g}\, \mathrm{d}^3\mathbf{x}. \tag{13}$$

Theorem 6. *Assume that the coefficient fields* (γ^μ, A) *satisfy the two admissibility conditions (2) and (9). In order that the Dirac Hamiltonian (11) be Hermitian for the scalar product (13), it is necessary and sufficient that*

$$\partial_0 \left(\sqrt{-g}\, A \gamma^0\right) = 0. \tag{14}$$

For DFW, hermiticity is not stable under a local similarity transformation

For DFW, *all* local similarity transformations are admissible, since condition (9) is always satisfied (with $A(X) \equiv A^\sharp$). In contrast, for TRD, condition (9) is quite demanding.

For DFW, in very general coordinates, the tetrad $(a^\mu{}_\alpha)$ may be chosen to satisfy $a^0{}_j = 0$. Taking for "flat" matrices the standard Dirac matrices $\gamma^{\sharp\,\mu}$, the hermiticity condition (14) then reduces to Leclerc's :

$$\partial_0(\sqrt{-g\,g^{00}}) = 0. \tag{15}$$

But, after a local similarity S, the condition (14) becomes

$$\partial_0(\sqrt{-g\,g^{00}}\,S^\dagger S) = 0, \tag{16}$$

which *cannot* be satisfied if (15) is, and if moreover $S^\dagger S = F(t)$.

Conclusion

- *Two* new gravitational Dirac eqs previously derived from wave mechanics. One obeys the equivalence principle, the other one has a preferred reference frame. Both see wave function as a *vector*. Usual (flat) QM unchanged.
- The three gravitational Dirac eqs (2+DFW) studied together.
- Current conservation asks for matrix eqn $D_\mu(A\gamma^\mu) = 0$.
- Hermiticity condition: $\partial_0 \left(\sqrt{-g}\, A\gamma^0\right) = 0$.
- For DFW, this is not stable under admissible similarity transformations: standard theory has a uniqueness Pb.