Non-uniqueness of the Dirac theory in a curved spacetime

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Context of this work

- Quantum effects in the classical gravitational field *are observed,* e.g. on neutrons: spin $\frac{1}{2}$ particles.
  $\Rightarrow$ Motivates work on the curved spacetime Dirac eq.

- Two alternative Dirac equations in a curved SpaceTime were derived, by directly applying the classical-quantum correspondence. (M.A.: *Found. Phys.* **38**, 1020–1045, 2008.)
  Thus, with the standard version (Fock & Weyl): **3** Dirac eqs!

- The basic quantum mechanics was studied for each of those three eqs (M.A.– F. Reifler, arXiv:0807.0570, gr-qc):
  - definition of the probability current & its conservation,
  - definition of the relevant scalar product,
  - Hamiltonian & its hermiticity.

Aim of this work

- Foregoing work: hermiticity of the Hamiltonian unstable under admissible changes of the coefficient fields! Means there is a non-uniqueness problem for the curved-spacetime Dirac eq.

- $\Rightarrow$ Present work: study the (non-)uniqueness of the Hamiltonian and energy operators, including the energy spectrum. Qualitative conclusions are the same for the three versions: *Non-uniqueness applies to altern. eqs too!*

Three Dirac equations in a curved spacetime

The 3 versions of the gravitational Dirac eq have the same form:

$$\gamma^\mu D_\mu \psi = -im\psi, \tag{1}$$

with $\gamma^\mu = \gamma^\mu(X)$ ($\mu = 0, ..., 3$) = field of 4×4 complex matrices defined over spacetime (S-T) $(\mathrm{V}, g_{\mu\nu})$, such that

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}\, \mathbf{1}_4, \quad \mu, \nu \in \{0, ..., 3\} \quad (\mathbf{1}_4 \equiv \mathrm{diag}(1,1,1,1)); \tag{2}$$

and where ψ is a *bispinor* field for standard eqn (Dirac- Fock-Weyl or **DFW**) but is a **4-vector** field for the two alternative eqs, based on the *tensor representation of the Dirac fields* (**TRD**);

and D_μ = covariant derivative, associated with a specific *connection*. For DFW: "spin connection", depends on (γ^μ) field.

Definition of the field of Dirac matrices

For DFW, one defines $\gamma^\mu = a^\mu{}_\alpha\, \gamma^{\sharp\alpha}$, with $u_\alpha = a^\mu{}_\alpha \partial_\mu$ an orthonormal tetrad field and $(\gamma^{\sharp\alpha})$ a set of "flat" Dirac matrices. One should be able to use *any* possible choice of $(\gamma^{\sharp\alpha})$. One should study the influence of both choices: $(\gamma^{\sharp\alpha})$ and (u_α).
For TRD, a tetrad field can also be used. Other possibilities exist.

To cope with any set (γ^μ): use the *hermitizing matrix* A. This is a 4×4 complex matrix such that

$$A^\dagger = A, \qquad (A\gamma^\mu)^\dagger = A\gamma^\mu \quad \mu = 0, ..., 3, \tag{3}$$

with $M^\dagger \equiv M^{*\,T}$ = Hermitian conjugate of matrix M. For usual choices $(\gamma^{\sharp\alpha})$ (Dirac, "chiral", Majorana), $\underline{A = \gamma^{\sharp 0}}$, constant.

We proved the existence of A (and that of B: for α^μ matrices).
(M.A. & F. Reifler: *Braz. J. Phys.* **38**, 248–258, 2008)

Local similarities

In a curved S-T $(\mathrm{V}, g_{\mu\nu})$, the Dirac matrices γ^μ and the hermitizing matrix A are fields, they depend on $X \in \mathrm{V}$.

If one changes from one field (γ^μ) to another one $(\tilde{\gamma}^\mu)$, the new field obtains by a *local similarity transformation* :

$$\exists S = S(X) \in \mathsf{GL}(4, \mathsf{C}) : \qquad \tilde{\gamma}^\mu(X) = S^{-1}\gamma^\mu(X)S, \quad \mu = 0, ..., 3. \tag{4}$$

Under a such change, the hermitizing matrix changes thus:

$$\tilde{A} = S^\dagger A S \ . \tag{5}$$

For the standard Dirac eq (DFW), the similarities are restricted to the spin group $\mathsf{Spin}(1,3)$, i.e., they are deduced from a local Lorentz transform $L(X)$ through the spinor representation.

The general Dirac Hamiltonian

Rewriting the Dirac eq (1) in the "Schrödinger" form:

$$i\frac{\partial\psi}{\partial t} = \mathrm{H}\psi, \qquad (t \equiv x^0), \tag{6}$$

gives the Hamiltonian operator:

$$\mathrm{H} \equiv m\alpha^0 - i\alpha^j D_j - i(D_0 - \partial_0), \tag{7}$$

with

$$\alpha^0 \equiv \gamma^0/g^{00}, \qquad \alpha^j \equiv \gamma^0\gamma^j/g^{00} \quad (j = 1, 2, 3). \tag{8}$$

Invariance condition of the Hamiltonian under a local similarity (DFW)

When does a local similarity $S(X)$, applied to the field of Dirac matrices γ^μ, leave H (eq (7)) invariant? I.e., when do we have

$$\tilde{\mathrm{H}} = S^{-1}\,\mathrm{H}\,S? \tag{9}$$

A straightforward calculation shows that we have (35) iff $S(X)$ is time-independent, $\partial_0 S = 0$. In the general case $g_{\mu\nu,0} \neq 0$, any possible field γ^μ depends on t : no way of finding a class of fields γ^μ exchanging with $\partial_0 S = 0$. I.e.: **The Dirac Hamiltonian is not unique.** (M.A.– F. Reifler, arXiv:0905.3686, gr-qc)

Note: For DFW, the spin connection matrices, $\Gamma_\mu \equiv D_\mu - \partial_\mu$, change after a similarity:

$$\tilde{\Gamma}_\mu = S^{-1}\Gamma_\mu S + S^{-1}(\partial_\mu S). \tag{10}$$

Invariance condition of the energy operator (DFW)

When the Hamiltonian H is not Hermitian, one should use the energy operator. Coincides with the Hermitian part of H:

$$\mathrm{E} = \mathrm{H} + \frac{i}{2\sqrt{-g}} B^{-1} \partial_0 \left(\sqrt{-g}\, B\right) = \frac{1}{2}\left(\mathrm{H} + \mathrm{H}^{\ddagger}\right), \quad B \equiv A\gamma^0. \qquad (11)$$

Again a straightforward calculation gives the invariance condition of E (for DFW):

$$B(\partial_0 S)S^{-1} - \left[B(\partial_0 S)S^{-1}\right]^{\dagger} \equiv 2\left[B(\partial_0 S)S^{-1}\right]^{a} = 0. \qquad (12)$$

Only very particular local similarities $S(X)$ do verify (12). Thus, there is a serious non-uniqueness problem for DFW (and for the alternative, "TRD" eqs as well). Could even the *spectrum* of E be non-unique? Let us see...

Explicit expression of the energy operator (DFW)

General expression of the change of E after a local similarity:

$$\delta \mathrm{E} \equiv S \tilde{\mathrm{E}} S^{-1} - \mathrm{E} = -iB^{-1} \left[B(\partial_0 S) S^{-1} \right]^a . \qquad (13)$$

We may select the tetrad for the starting (untilded) fields such that ${a^0}_j = 0$ (this is standard anyway), whence (28), thus

$$\delta \mathrm{E} = -i \left[(\partial_0 S) S^{-1} \right]^a . \qquad (14)$$

$(\partial_0 S) S^{-1}$ = generic element of $\mathcal{G}$, Lie algebra of $\mathsf{Spin}(1,3)$ – whose the $s^{\alpha\beta} \equiv [\gamma^{\sharp\alpha}, \gamma^{\sharp\beta}]$ $(\alpha < \beta)$ make a basis. Hence

$$\delta \mathrm{E} = -i \left[\omega_{\alpha\beta} s^{\alpha\beta} \right]^a = -i \sum_{j,k=1}^{3} \omega_{jk} s^{jk}, \qquad (15)$$

and, depending on the local Lorentz $L(X)$ that defines $S(X) = \mathsf{S}(L(X))$, the 6 coeffs $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ depend arbitrarily on $X \in \mathrm{V}$.

The case with the"chiral" Dirac matrices

If the "flat" Dirac matrices $\gamma^{\sharp\alpha}$ are the "chiral" ones, we get

$$\delta\mathrm{E} = -i \sum_{j,k=1}^{3} \omega_{jk} s^{jk} = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}, \qquad N \equiv -\frac{1}{2}\vec{\theta}.\vec{\sigma} \qquad (16)$$

where $\vec{\theta} \equiv (\theta^k)$ with $\theta^1 \equiv \omega_{23}$ (circular), and where $\vec{\sigma} \equiv (\sigma_k)$ with σ_k = Pauli matrices.

Depending on the 3 real numbers $\omega_{jk},\ 1 \leq j < k \leq 3$, the matrix N can be *any* Hermitian matrix 2×2 with zero trace. Any such matrix has 2 eigenvalues $\mu \in \mathsf{R}$ and $-\mu$, and has an orthonormal basis of eigenvectors: respectively $u \in \mathsf{C}^2$ for μ, and v for $-\mu$.

Non-uniqueness of the energy spectrum (DFW)

A small perturbation: $S(\varepsilon, X) = I + \varepsilon\,(\delta S)(X) + O(\varepsilon^2)$, modifies each eigenvalue of E : $\delta\lambda = (\psi \mid \delta \mathrm{E}(\varepsilon)\psi) + O(\varepsilon^2)$ with ψ the eigenfunction for the unperturbed state. With (16), and decomposing: $\psi = (\phi, \chi)$, we find:

$$\delta\lambda = \int \psi^\dagger \delta\mathrm{E}\,\psi\,\underline{\sqrt{-g\,g^{00}}\,\mathrm{d}^3\mathbf{x}} = \int (\phi^\dagger N \phi + \chi^\dagger N \chi)\,\underline{\mathrm{dV}}. \qquad (17)$$

Fix $\mu > 0$ and t. $\forall x$ in the space manifold M, let $N = N(x)$ be such that $\phi(x)$ be the eigenvector of $N(x)$ for the eigenvalue μ, whence

$$\phi^\dagger N \phi = \mu\,\phi^\dagger\phi, \quad \chi^\dagger N \chi \geq -\mu\,\chi^\dagger\chi. \qquad (18)$$

Here $\geq$ becomes $=$ only if $\chi(x) \perp \phi(x)$. So $\delta\lambda > 0$ unless if i) $\chi(x) \perp \phi(x)$ a.e. *and* ii) $\int \phi^\dagger\phi\,\mathrm{dV} = \int \chi^\dagger\chi\,\mathrm{dV}$. Rare ! i) $\Rightarrow J^\mu$ light-like a.e., *impossible* if $m > 0$. Q.E.D.

Conclusion

- The 3 gravitational Dirac eqs (standard: DFW, 2 alternative: TRD) were studied together, using the hermitizing matrix A.
- The Hamiltonian operator H is not unique: it depends on the admissible choice of the field of Dirac matrices. Idem for the energy operator E. True for DFW and for TRD.
- The *spectrum* of E is itself non-unique. All of these results apply already to a *flat* spacetime in a non-inertial frame.

The classical energy and its frame dependence

In GR, there is no covariant concept of local energy for the *fields* (cf. energy-momentum *pseudo*-tensor). But, for a *test particle*, in any arbitrary reference frame F, there is a well-defined *Hamiltonian energy* (it depends on F & on time):

* Geodesic motion in the Lorentzian manifold $(\mathrm{V}, g_{\mu\nu})$ derives from the ("super-")Hamiltonian over *8-dimensional* phase space: $\tilde{H}[(p_\mu),(x^\nu)] \equiv \text{kinetic energy} \equiv \frac{1}{2} g^{\mu\nu}((x^\rho)) p_\mu p_\nu$. ($c = 1$.) (Cf. Arnold.) Note that $\tilde{H}$ does not depend on (proper) time τ.

* *Hence,* in any given coordinate system, we deduce a "normal" Hamiltonian over *6-dimensional* phase space, by *dimensional reduction*: $H[(p_j),(x^j), t \equiv x^0] \equiv p_0$ extracted from $g^{\mu\nu} p_\mu p_\nu - m^2 = 0$. (Cf. Arnold.) $E = H \equiv p_0$ is *invariant* under *spatial* coordinate changes $x'^0 = x^0,\ x'^j = f^j((x^k))$.

Definition of the probability current

The probability current is defined as

$$J^\mu = \psi^\dagger A \gamma^\mu \psi. \tag{19}$$

This is generally-covariant: J^μ is a *4-vector,* for DFW and for TRD as well. In a curved S-T $(\mathrm{V}, g_{\mu\nu})$, γ^μ and A depend on $X \in \mathrm{V}$.

The current (19) is *independent of the choice of the Dirac matrices:* Under a local similarity

$$\exists S = S(X) \in \mathsf{GL}(4, \mathsf{C}) : \qquad \tilde{\gamma}^\mu(X) = S^{-1}\gamma^\mu(X)S, \quad \mu = 0, ..., 3, \tag{20}$$

$$\tilde{A} = S^\dagger A S \ . \tag{21}$$

If we change simultaneously $\tilde{\psi} = S^{-1}\psi$, we get indeed $\tilde{J}^\mu = J^\mu$.

Condition for current conservation

Theorem 1. *Consider the general Dirac equation in a curved spacetime (1), thus either DFW or any of the two TRD equations. In order that any* ψ *solution of (1) satisfy the current conservation*

$$D_\mu J^\mu = 0, \tag{22}$$

it is necessary and sufficient that

$$D_\mu (A\gamma^\mu) = 0. \tag{23}$$

Corollary 1. *For DFW theory, the hermitizing matrix field* $A(X)$ *can be imposed to be the constant* matrix $A^\sharp$*, i.e., a hermitizing matrix for the "flat" matrices* $\gamma^{\sharp\alpha}$ *such that* $\gamma^\mu = a^\mu{}_\alpha\, \gamma^{\sharp\alpha}$*. Then the current conservation applies to any solution of the DFW equation.* (M.A.– F. Reifler, arXiv:0807.0570, gr-qc)

The Hamiltonian is frame dependent

Hamiltonian of the Dirac eq (1) :

$$\mathrm{H} \equiv m\alpha^0 - i\alpha^j D_j - i(D_0 - \partial_0), \tag{24}$$

with

$$\alpha^0 \equiv \gamma^0/g^{00}, \qquad \alpha^j \equiv \gamma^0\gamma^j/g^{00} \quad (j = 1,2,3). \tag{25}$$

In order that the Hamiltonians H and H', before and after a coordinate change, be equivalent operators, the coordinate change must be **spatial**: $x'^0 = x^0,\ x'^j = f^j((x^k))$. Then, both sides of the Schrödinger eq (6) behave as scalars for DFW, and as vectors for TRD: **H depends on the reference frame** (3D congruence of world lines) considered. A general fact.

Hermiticity condition of the Hamiltonian

Theorem 5. *A* necessary *condition for the scalar product of time-independent wave functions to be time independent and for the Hamiltonian* H *to be a Hermitian operator, is that the scalar product should be*

$$(\psi \mid \varphi) \equiv \int_{\mathrm{R}^3} \psi^\dagger A \gamma^0 \varphi \ \sqrt{-g}\, \mathrm{d}^3\mathbf{x}. \tag{26}$$

Theorem 6. *Assume that the coefficient fields* (γ^μ, A) *satisfy the two admissibility conditions (2) (and (23), for TRD). In order that the Dirac Hamiltonian (7) be Hermitian for the scalar product (26), it is necessary and sufficient that*

$$\partial_0 \left(\sqrt{-g}\, A \gamma^0 \right) = 0. \tag{27}$$

Problem: Condition is unstable under local similarity transforms!!

Unstability of Hermiticity: DFW case

For DFW, all local similarities S with $\forall X \quad S(X) \in \mathsf{Spin}(1,3)$ are admissible, since condition (23) is always satisfied (with the choice $A(X) \equiv A^\sharp$: see Corollary 1). Moreover, in very general coordinates, the tetrad $(a^\mu{}_\alpha)$ may be chosen to satisfy $a^0{}_j = 0$. Then $a^0{}_0 = \sqrt{g^{00}}$ from the orthonormality of the tetrad. Take for "flat" matrices $\gamma^{\sharp\,\alpha}$ standard Dirac matrices, for which $A = \gamma^{\sharp 0}$. Thus

$$B \equiv A\,\gamma^0 = \gamma^{\sharp 0}(a^0{}_0\gamma^{\sharp 0}) = \sqrt{g^{00}}\,\mathbf{1}_4. \tag{28}$$

The hermiticity condition (27) then reduces to Leclerc's (2006):

$$\partial_0(\sqrt{-g\,g^{00}}) = 0. \tag{29}$$

But, after a local similarity S, the condition (27) becomes

$$\partial_0(\sqrt{-g\,g^{00}}\;S^\dagger S) = 0, \tag{30}$$

which *cannot* be satisfied if (29) is, and if moreover $S^\dagger S = F(t)$.

Definition of equivalent operators

With each of 2 coefficient fields: (γ^μ, A) and $(\tilde{\gamma}^\mu, \tilde{A})$, corresponds a unique scalar product. These two scalar products are *isometrically equivalent* through $\psi \mapsto \tilde{\psi} \equiv S^{-1}\psi$:

$$(\tilde{\psi} \,\tilde{|}\, \tilde{\varphi}) \equiv \int_{\mathrm{R}^3} (S^{-1}\psi)^\dagger \, S^\dagger B S \, (S^{-1}\varphi)\sqrt{-g}\, \mathrm{d}^3\mathbf{x} = (\psi \mid \varphi). \qquad (31)$$

H is fully determined by the set of the products $(\mathrm{H}\,\psi \mid \varphi)$, for $\psi, \varphi \in \mathcal{D} \equiv \mathrm{Dom}(\mathrm{H})$.
Thus, H and $\tilde{\mathrm{H}}$ are *equivalent* iff, for all $\psi, \varphi \in \mathcal{D}$, we have

$$(\tilde{\mathrm{H}}\,\tilde{\psi} \,\tilde{|}\, \tilde{\varphi}) = (\mathrm{H}\,\psi \mid \varphi). \qquad (32)$$

But, from (31), we get directly:

$$(\widetilde{\mathrm{H}\,\psi} \,\tilde{|}\, \tilde{\varphi}) = (\mathrm{H}\,\psi \mid \varphi). \qquad (33)$$

Hence, in order that H and $\tilde{\mathrm{H}}$ be equivalent operators, it is necessary and sufficient that, for all $\psi \in \mathcal{D}$, we have

$$\widetilde{\mathrm{H}\,\psi} = \tilde{\mathrm{H}}\,\tilde{\psi}. \tag{34}$$

Since, from $\tilde{\psi} \equiv S^{-1}\psi$, we have $\widetilde{\mathrm{H}\,\psi} \equiv S^{-1}\,\mathrm{H}\psi \equiv S^{-1}\,\mathrm{H}S\tilde{\psi}$, this rewrites as

$$\tilde{\mathrm{H}} = S^{-1}\,\mathrm{H}\,S. \tag{35}$$

This is the condition of equivalence of the Dirac Hamiltonians associated with two different choices of the coefficient fields. (Idem for the energy operator E.)

Transformation of Dirac wave function

Consider Minkowski ST and ask that, after linear coordinate changes $L \in \mathrm{G}$, with G a linear group, the Dirac wave function ψ become

$$\psi'(X') = S.\psi(X), \qquad S = \mathsf{S}(L), \tag{36}$$

for some operator function S of L. S has to be a representation $\mathrm{G} \to \mathsf{GL}(4, \mathsf{C})$. The Dirac eq. of special relativity (SR) becomes

$$(i\gamma'^{\nu}\, \partial'_{\nu} - m)\psi' = 0, \quad \gamma'^{\nu} \equiv L^{\nu}_{\mu} S \gamma^{\mu} S^{-1}. \tag{37}$$

Standard statement: *Relativity asks that* $\gamma'^{\nu} = \gamma^{\nu}$ (whence the spinor representation). *But no!* Archetypically relativistic is the eq of motion for a particle with 4-velocity U^{μ} in e.m. field F^{μ}_{ν}:

$$m\frac{\mathrm{d}U^{\mu}}{\mathrm{d}s} = qF^{\mu}_{\nu}U^{\nu}, \qquad \text{or} \quad m\frac{\mathrm{d}U}{\mathrm{d}s} = qFU. \tag{38}$$

Here, matrix $F \equiv (F^{\mu}_{\nu})$ is *not* invariant: $F' = LFL^{-1} \neq F$.

4-vector transformation of Dirac wave function

The simplest possibility for S is the identity: $\mathsf{S}(L) = L$, thus the *4-vector transformation of the Dirac wave function:*

$$\psi'(X') = L.\psi(X), \qquad \text{or} \quad \psi'^{\mu} = L^{\mu}_{\nu}\psi^{\nu}. \tag{39}$$

Then, the Dirac matrices transform in the following way:

$$\gamma'^{\mu} \equiv L^{\mu}_{\nu} L \gamma^{\nu} L^{-1}, \tag{40}$$

which means that *the components* $(\gamma^{\mu})^{\rho}_{\nu}$ *form a* $\binom{2}{1}$ *tensor.*

- The anticommutation is preserved, $[\gamma'^{\mu}, \gamma'^{\nu}]_{+} = 2g'^{\mu\nu}\, 1$.
- Direct physical consequences of the Dirac eq unchanged: the explicit *equation*, hence its *solutions*, stay unchanged. (In SR, the choice of the constant set (γ^{μ}) has no effect on QM quantities: M.A. & F. Reifler, *Braz. J. Phys.* **38**, 248–258, 2008.) Transform$^{\mathrm{n}}$ (39)–(40) also usable for curved ST.