Non-uniqueness of the Dirac theory in a curved spacetime

Mayeul Arminjon\textsuperscript{1} and Frank Reifler\textsuperscript{2}

\textsuperscript{1} CNRS (Section of Theoretical Physics)
Lab ‘Soils, Solids, Structures–Risks’, Grenoble, France

\textsuperscript{2} Lockheed Martin Corporation,
Moorestown, New Jersey, USA

MCCQG, Kolymbari, September 14–18, 2009
Context of this work

- Quantum effects in the classical gravitational field are observed, e.g. on neutrons: spin $\frac{1}{2}$ particles.
  $\Rightarrow$ Motivates work on the curved spacetime Dirac eq.

- Two alternative Dirac equations in a curved SpaceTime were derived, by directly applying the classical-quantum correspondence. (M.A.: Found. Phys. 38, 1020–1045, 2008.)
  Thus, with the standard version (Fock & Weyl): 3 Dirac eqs!

- The basic quantum mechanics was studied for each of those three eqs (M.A.– F. Reifler, arXiv:0807.0570, gr-qc):
  - definition of the probability current & its conservation,
  - definition of the relevant scalar product,
  - Hamiltonian & its hermiticity.
Aim of this work

- Foregoing work: hermiticity of the Hamiltonian unstable under admissible changes of the coefficient fields! Means there is a non-uniqueness problem for the curved-spacetime Dirac eq.

- Present work: study the (non-)uniqueness of the Hamiltonian and energy operators, including the energy spectrum. Qualitative conclusions are the same for the three versions: Non-uniqueness applies to altern. eqs too!
Three Dirac equations in a curved spacetime

The 3 versions of the gravitational Dirac eq have the same form:

$$\gamma^\mu D_\mu \psi = -im\psi,$$

(1)

with $\gamma^\mu = \gamma^\mu(X)$ ($\mu = 0, \ldots, 3$) = field of $4 \times 4$ complex matrices defined over spacetime (S-T) $(V, g_{\mu\nu})$, such that

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} 1_4, \quad \mu, \nu \in \{0, \ldots, 3\} \quad (1_4 \equiv \text{diag}(1, 1, 1, 1));$$

(2)

and where $\psi$ is a bispinor field for standard eqn (Dirac- Fock-Weyl or DFW) but is a 4-vector field for the two alternative eqs, based on the tensor representation of the Dirac fields (TRD);

and $D_\mu = \text{covariant derivative}$, associated with a specific connection. For DFW: “spin connection”, depends on $(\gamma^\mu)$ field.
Definition of the field of Dirac matrices

For DFW, one defines $\gamma^\mu = a^{\mu\alpha} \gamma^{#\alpha}$, with $u_\alpha = a^{\mu\alpha} \partial_\mu$ an orthonormal tetrad field and $(\gamma^{#\alpha})$ a set of “flat” Dirac matrices. One should be able to use any possible choice of $(\gamma^{#\alpha})$. One should study the influence of both choices: $(\gamma^{#\alpha})$ and $(u_\alpha)$. For TRD, a tetrad field can also be used. Other possibilities exist.

To cope with any set $(\gamma^\mu)$: use the hermitizing matrix $A$. This is a $4 \times 4$ complex matrix such that

$$A^\dagger = A, \quad (A \gamma^\mu)^\dagger = A \gamma^\mu \quad \mu = 0, \ldots, 3, \quad (3)$$

with $M^\dagger \equiv M^* T = \text{Hermitian conjugate of matrix } M$. For usual choices $(\gamma^{#\alpha})$ (Dirac, “chiral”, Majorana), $A = \gamma^{#0}$, constant.

We proved the existence of $A$ (and that of $B$: for $\alpha^{\mu}$ matrices). (M.A. & F. Reifler: *Braz. J. Phys.* 38, 248–258, 2008)
Local similarities

In a curved S-T $\left(V, g_{\mu\nu}\right)$, the Dirac matrices $\gamma^\mu$ and the hermitizing matrix $A$ are fields, they depend on $X \in V$.

If one changes from one field $(\gamma^\mu)$ to another one $(\tilde{\gamma}^\mu)$, the new field obtains by a local similarity transformation:

$$\exists S = S(X) \in \text{GL}(4, \mathbb{C}) : \quad \tilde{\gamma}^\mu(X) = S^{-1}\gamma^\mu(X)S, \quad \mu = 0, \ldots, 3.$$  \hspace{1cm} (4)

Under a such change, the hermitizing matrix changes thus:

$$\tilde{A} = S^\dagger AS.$$  \hspace{1cm} (5)

For the standard Dirac eq (DFW), the similarities are restricted to the spin group $\text{Spin}(1, 3)$, i.e., they are deduced from a local Lorentz transform $L(X)$ through the spinor representation.
The general Dirac Hamiltonian

Rewriting the Dirac eq (1) in the “Schrödinger” form:

\[ i \frac{\partial \psi}{\partial t} = H \psi, \quad (t \equiv x^0), \]  

(6)

gives the Hamiltonian operator:

\[ H \equiv m\alpha^0 - i\alpha^j D_j - i(D_0 - \partial_0), \]  

(7)

with

\[ \alpha^0 \equiv \gamma^0 / g^{00}, \quad \alpha^j \equiv \gamma^0 \gamma^j / g^{00} \quad (j = 1, 2, 3). \]  

(8)
Invariance condition of the Hamiltonian under a local similarity (DFW)

When does a local similarity \( S(X) \), applied to the field of Dirac matrices \( \gamma^\mu \), leave \( H \) (eq (7)) invariant? I.e., when do we have

\[
\tilde{H} = S^{-1} H S?
\]  

(9)

A straightforward calculation shows that we have (35) iff \( S(X) \) is time-independent, \( \partial_0 S = 0 \). In the general case \( g_{\mu\nu,0} \neq 0 \), any possible field \( \gamma^\mu \) depends on \( t \) : no way of finding a class of fields \( \gamma^\mu \) exchanging with \( \partial_0 S = 0 \). I.e.: The Dirac Hamiltonian is not unique. (M.A.– F. Reifler, arXiv:0905.3686, gr-qc)

Note: For DFW, the spin connection matrices, \( \Gamma_\mu \equiv D_\mu - \partial_\mu \), change after a similarity:

\[
\tilde{\Gamma}_\mu = S^{-1} \Gamma_\mu S + S^{-1} (\partial_\mu S).
\]  

(10)
Invariance condition of the energy operator (DFW)

When the Hamiltonian $H$ is not Hermitian, one should use the energy operator. Coincides with the Hermitian part of $H$:

$$E = H + \frac{i}{2\sqrt{-g}} B^{-1} \partial_0 (\sqrt{-g} B) = \frac{1}{2} (H + H^\dagger), \quad B \equiv A\gamma^0. \quad (11)$$

Again a straightforward calculation gives the invariance condition of $E$ (for DFW):

$$B(\partial_0 S) S^{-1} - [B(\partial_0 S) S^{-1}]^\dagger \equiv 2 [B(\partial_0 S) S^{-1}]^a = 0. \quad (12)$$

Only very particular local similarities $S(X)$ do verify $\text{(12)}$. Thus, there is a serious non-uniqueness problem for DFW (and for the alternative, “TRD” eqs as well). Could even the spectrum of $E$ be non-unique? Let us see...
Explicit expression of the energy operator (DFW)

General expression of the change of $E$ after a local similarity:

$$\delta E \equiv S \tilde{E} S^{-1} - E = -i B^{-1} \left[ B(\partial_0 S) S^{-1} \right]^a. \quad (13)$$

We may select the tetrad for the starting (untilded) fields such that $a^0_j = 0$ (this is standard anyway), whence (28), thus

$$\delta E = -i \left[ (\partial_0 S) S^{-1} \right]^a. \quad (14)$$

$$(\partial_0 S) S^{-1} = \text{generic element of } G, \text{ Lie algebra of } Spin(1, 3) -$$

whose the $s^{\alpha\beta} \equiv [\gamma^\#\alpha, \gamma^\#\beta]$ ($\alpha < \beta$) make a basis. Hence

$$\delta E = -i \left[ \omega_{\alpha\beta} s^{\alpha\beta} \right]^a = -i \sum_{j,k=1}^3 \omega_{jk} s^{jk}, \quad (15)$$

and, depending on the local Lorentz $L(X)$ that defines $S(X) = S(L(X))$, the 6 coeffs $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ depend arbitrarily on $X \in V$. 
The case with the “chiral” Dirac matrices

If the “flat” Dirac matrices $\gamma^{\#\alpha}$ are the “chiral” ones, we get

$$\delta E = -i \sum_{j,k=1}^{3} \omega_{jk} s^{jk} = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}, \quad N \equiv -\frac{1}{2} \vec{\theta}.\vec{\sigma} \quad (16)$$

where $\vec{\theta} \equiv (\theta^k)$ with $\theta^1 \equiv \omega_{23}$ (circular), and where $\vec{\sigma} \equiv (\sigma^k)$ with $\sigma^k = \text{Pauli matrices}.$

Depending on the 3 real numbers $\omega_{jk}, 1 \leq j < k \leq 3$, the matrix $N$ can be any Hermitian matrix $2 \times 2$ with zero trace. Any such matrix has 2 eigenvalues $\mu \in \mathbb{R}$ and $-\mu$, and has an orthonormal basis of eigenvectors: respectively $u \in \mathbb{C}^2$ for $\mu$, and $v$ for $-\mu.$
Non-uniqueness of the energy spectrum (DFW)

A small perturbation: \( S(\varepsilon, X) = I + \varepsilon (\delta S)(X) + O(\varepsilon^2) \), modifies each eigenvalue of \( E : \delta \lambda = (\psi | \delta E(\varepsilon)\psi) + O(\varepsilon^2) \) with \( \psi \) the eigenfunction for the unperturbed state. With (16), and decomposing: \( \psi = (\phi, \chi) \), we find:

\[
\delta \lambda = \int \psi^\dagger \delta E \psi \sqrt{-g} g^{00} \, d^3x = \int (\phi^\dagger N\phi + \chi^\dagger N\chi) \, dV. \tag{17}
\]

Fix \( \mu > 0 \) and \( t \). \( \forall x \) in the space manifold \( M \), let \( N = N(x) \) be such that \( \phi(x) \) be the eigenvector of \( N(x) \) for the eigenvalue \( \mu \), whence

\[
\phi^\dagger N\phi = \mu \phi^\dagger \phi, \quad \chi^\dagger N\chi \geq -\mu \chi^\dagger \chi. \tag{18}
\]

Here \( \geq \) becomes \( = \) only if \( \chi(x) \bot \phi(x) \). So \( \delta \lambda > 0 \) unless if i) \( \chi(x) \bot \phi(x) \) a.e. and ii) \( \int \phi^\dagger \phi \, dV = \int \chi^\dagger \chi \, dV \). Rare ! i) \( \Rightarrow J^\mu \) light-like a.e., impossible if \( m > 0 \). Q.E.D.
Conclusion

- The 3 gravitational Dirac eqs (standard: DFW, 2 alternative: TRD) were studied together, using the hermitizing matrix $A$.

- The Hamiltonian operator $H$ is not unique: it depends on the admissible choice of the field of Dirac matrices. Idem for the energy operator $E$. True for DFW and for TRD.

- The spectrum of $E$ is itself non-unique. All of these results apply already to a flat spacetime in a non-inertial frame.
The classical energy and its frame dependence

In GR, there is no covariant concept of local energy for the fields (cf. energy-momentum pseudo-tensor). But, for a test particle, in any arbitrary reference frame $F$, there is a well-defined Hamiltonian energy (it depends on $F$ & on time):

* Geodesic motion in the Lorentzian manifold $(V, g_{\mu\nu})$ derives from the ("super-")Hamiltonian over 8-dimensional phase space: $\tilde{H}[(p_\mu), (x^\nu)] \equiv \text{kinetic energy} \equiv \frac{1}{2} g^{\mu\nu} (x^0)p_\mu p_\nu. (c = 1.)$ (Cf. Arnold.) Note that $\tilde{H}$ does not depend on (proper) time $\tau$.

* Hence, in any given coordinate system, we deduce a "normal" Hamiltonian over 6-dimensional phase space, by dimensional reduction: $H[(p_j), (x^j), t \equiv x^0] \equiv p_0$ extracted from $g^{\mu\nu} p_\mu p_\nu - m^2 = 0.$ (Cf. Arnold.) $E = H \equiv p_0$ is invariant under spatial coordinate changes $x'^0 = x^0$, $x'^j = f^j ((x^k))$. 
Definition of the probability current

The probability current is defined as

$$J^\mu = \psi^\dagger A\gamma^\mu \psi.$$  (19)

This is generally-covariant: $J^\mu$ is a 4-vector, for DFW and for TRD as well. In a curved S-T $(V, g_{\mu\nu})$, $\gamma^\mu$ and $A$ depend on $X \in V$.

The current (19) is independent of the choice of the Dirac matrices: Under a local similarity

$$\exists S = S(X) \in \text{GL}(4, \mathbb{C}) : \tilde{\gamma}^\mu(X) = S^{-1}\gamma^\mu(X)S, \quad \mu = 0, \ldots, 3,$$

$$\tilde{A} = S^\dagger AS.$$  (20)

If we change simultaneously $\tilde{\psi} = S^{-1}\psi$, we get indeed $\tilde{J}^\mu = J^\mu$.  (21)
**Condition for current conservation**

**Theorem 1.** Consider the general Dirac equation in a curved spacetime (1), thus either DFW or any of the two TRD equations. In order that any $\psi$ solution of (1) satisfy the current conservation

$$D_\mu J^\mu = 0,$$

(22)

it is necessary and sufficient that

$$D_\mu (A\gamma^\mu) = 0.$$  

(23)

**Corollary 1.** For DFW theory, the hermitizing matrix field $A(X)$ can be imposed to be the constant matrix $A^\#$, i.e., a hermitizing matrix for the “flat” matrices $\gamma^\#_\alpha$ such that $\gamma^\mu = a^\mu_\alpha \gamma^\#_\alpha$. Then the current conservation applies to any solution of the DFW equation. (M.A.– F. Reifler, arXiv:0807.0570, gr-qc)
The Hamiltonian is frame dependent

Hamiltonian of the Dirac eq (1):

\[ H \equiv m\alpha^0 - i\alpha^j D_j - i(D_0 - \partial_0), \]  

(24)

with

\[ \alpha^0 \equiv \gamma^0 / g^{00}, \quad \alpha^j \equiv \gamma^0 \gamma^j / g^{00} \quad (j = 1, 2, 3). \]  

(25)

In order that the Hamiltonians \( H \) and \( H' \), before and after a coordinate change, be equivalent operators, the coordinate change must be spatial: \( x'^0 = x^0, \quad x'^j = f^j ((x^k)) \). Then, both sides of the Schrödinger eq (6) behave as scalars for DFW, and as vectors for TRD: \( H \) depends on the reference frame (3D congruence of world lines) considered. A general fact.
**Hermiticity condition of the Hamiltonian**

**Theorem 5.** A necessary condition for the scalar product of time-independent wave functions to be time independent and for the Hamiltonian $H$ to be a Hermitian operator, is that the scalar product should be

$$
(\psi | \varphi) \equiv \int_{\mathbb{R}^3} \psi^\dagger A\gamma^0 \varphi \sqrt{-g} \, d^3 x.
$$

(26)

**Theorem 6.** Assume that the coefficient fields $(\gamma^\mu, A)$ satisfy the two admissibility conditions (2) (and (23), for TRD). In order that the Dirac Hamiltonian (7) be Hermitian for the scalar product (26), it is necessary and sufficient that

$$
\partial_0 (\sqrt{-g} A\gamma^0) = 0.
$$

(27)

**Problem:** Condition is unstable under local similarity transforms!!
Unstability of Hermiticity: DFW case

For DFW, all local similarities $S$ with $\forall X \ S(X) \in \text{Spin}(1, 3)$ are admissible, since condition (23) is always satisfied (with the choice $A(X) \equiv A^\#$: see Corollary 1). Moreover, in very general coordinates, the tetrad $(a^\mu_\alpha)$ may be chosen to satisfy $a^0_j = 0$. Then $a^0_0 = \sqrt{g^{00}}$ from the orthonormality of the tetrad. Take for “flat” matrices $\gamma^\#_\alpha$ standard Dirac matrices, for which $A = \gamma^\#_0$. Thus

$$B \equiv A \gamma^0 = \gamma^\#_0(a^0_0 \gamma^\#_0) = \sqrt{g^{00}} \ 1_4. \quad (28)$$

The hermiticity condition (27) then reduces to Leclerc’s (2006):

$$\partial_0 (\sqrt{-g g^{00}}) = 0. \quad (29)$$

But, after a local similarity $S$, the condition (27) becomes

$$\partial_0 (\sqrt{-g g^{00}} S^\dagger S) = 0, \quad (30)$$

which cannot be satisfied if (29) is, and if moreover $S^\dagger S = F(t)$. 


Definition of equivalent operators

With each of 2 coefficient fields: \((\gamma^\mu, A)\) and \((\tilde{\gamma}^\mu, \tilde{A})\), corresponds a unique scalar product. These two scalar products are \textit{isometrically equivalent} through \(\psi \mapsto \tilde{\psi} \equiv S^{-1}\psi\):

\[
(\tilde{\psi} | \tilde{\varphi}) \equiv \int_{\mathbb{R}^3} (S^{-1}\psi)^\dagger S^\dagger BS (S^{-1}\varphi) \sqrt{-g} \, d^3x = (\psi | \varphi) .
\] (31)

H is fully determined by the set of the products \((H \psi | \varphi)\), for \(\psi, \varphi \in \mathcal{D} \equiv \text{Dom}(H)\).

Thus, \(H\) and \(\tilde{H}\) are \textit{equivalent} iff, for all \(\psi, \varphi \in \mathcal{D}\), we have

\[
(\tilde{H} \tilde{\psi} | \tilde{\varphi}) = (H \psi | \varphi) .
\] (32)

But, from (31), we get directly:

\[
(\tilde{H} \tilde{\psi} | \tilde{\varphi}) = (H \psi | \varphi) .
\] (33)
Hence, in order that $H$ and $\tilde{H}$ be equivalent operators, it is necessary and sufficient that, for all $\psi \in \mathcal{D}$, we have

$$\tilde{H}\psi = \tilde{H}\tilde{\psi}. \quad (34)$$

Since, from $\tilde{\psi} \equiv S^{-1}\psi$, we have $\tilde{H}\psi \equiv S^{-1}H\psi \equiv S^{-1}HS\tilde{\psi}$, this rewrites as

$$\tilde{H} = S^{-1}HS. \quad (35)$$

This is the condition of equivalence of the Dirac Hamiltonians associated with two different choices of the coefficient fields. (Idem for the energy operator $E$.)
**Transformation of Dirac wave function**

Consider Minkowski ST and ask that, after linear coordinate changes $L \in G$, with $G$ a linear group, the Dirac wave function $\psi$ become

$$
\psi'(X') = S \psi(X), \quad S = S(L),
$$

(36)

for some operator function $S$ of $L$. $S$ has to be a representation $G \to GL(4, \mathbb{C})$. The Dirac eq. of special relativity (SR) becomes

$$
(i \gamma'^{\nu} \partial'_\nu - m)\psi' = 0, \quad \gamma'^{\nu} \equiv L_\mu^\nu S \gamma^{\mu} S^{-1}.
$$

(37)

Standard statement: Relativity asks that $\gamma'^{\nu} = \gamma^{\nu}$ (whence the spinor representation). But no! Archetypically relativistic is the eq of motion for a particle with 4-velocity $U^\mu$ in e.m. field $F^\mu_\nu$:

$$
m \frac{dU^\mu}{ds} = q F^\mu_\nu U^\nu, \quad \text{or} \quad m \frac{dU}{ds} = q FU.
$$

(38)

Here, matrix $F \equiv (F^\mu_\nu)$ is not invariant: $F' = LFL^{-1} \neq F$. 

4-vector transformation of Dirac wave function

The simplest possibility for $S$ is the identity: $S(L) = L$, thus the 4-vector transformation of the Dirac wave function:

$$\psi'(X') = L \psi(X), \quad \text{or} \quad \psi'^\mu = L_\nu^\mu \psi^\nu. \quad (39)$$

Then, the Dirac matrices transform in the following way:

$$\gamma'^\mu \equiv L_\nu^\mu L^\nu L^{-1}, \quad (40)$$

which means that the components $(\gamma^\mu)_\rho^\nu$ form a $\left( \begin{array}{c} 2 \\ 1 \end{array} \right)$ tensor.

- The anticommutation is preserved, $[\gamma'^\mu, \gamma'^\nu]_+ = 2g'^\mu_\nu \mathbb{1}$.
- Direct physical consequences of the Dirac eq unchanged: the explicit equation, hence its solutions, stay unchanged. (In SR, the choice of the constant set $(\gamma^\mu)$ has no effect on QM quantities: M.A. & F. Reifler, Braz. J. Phys. 38, 248–258, 2008.) Transform $n$ (39)–(40) also usable for curved ST.