Motion of a test particle according to the scalar ether theory of gravitation and application to its celestial mechanics

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Short title: Motion in the scalar ether theory of gravitation

Abstract

The standard interpretations of special relativity (Einstein-Minkowski) and general relativity (GR) lead to a drastically changed notion of time: the eternalism or block universe theory. This has strong consequences for our thinking about time and for the development of new fundamental theories. It is therefore important to check this thoroughly. The Lorentz-Poincaré interpretation, which sees the relativistic effects as following from a "true" Lorentz contraction of all objects in their motion through the ether, uses a conservative concept of time and is in the absence of gravitation indistinguishable from the standard interpretation; but there exists currently no accepted gravitation theory for it. The scalar ether theory of gravitation is a candidate for such a theory; it is presented and discussed. The equations of motion for a test particle are derived; the case of a uniformly moving massive body is discussed and then specialized to the case of spherical symmetry. Formulas for the acceleration of test particles are given in the preferred frame of the ether and in the rest frame of the massive body that moves with velocity $\mathbf{V}$ with respect to the ether. When the body rests in the ether ($\mathbf{V} = \mathbf{0}$), the acceleration is up to order c^{-2} identical to GR. The acceleration of a test particle for $\mathbf{V} \neq \mathbf{0}$ is given; this makes it possible to fit observations in celestial mechanics to ephemerides with **V** as a free parameter. The current status of such fits (though to ephemerides and not to observations) is presented and discussed.

Keywords: alternative theories of gravitation; Lorentzian metric; preferred reference frame; test particle; celestial mechanics.

1 Introduction and summary

There are two empirically indistinguishable interpretations of special relativity: the Lorentz-Poincaré interpretation, which sees the relativistic effects as following from the "true" Lorentz contraction of all objects in their motion through the ether, and the standard (Einstein-Minkowski) interpretation [1, 2]. The latter is currently preferred for various reasons. An important argument is that general relativity (GR) is an extension of the standard (Einstein-Minkowski) relativity and is not compatible with the Lorentz-Poincaré interpretation [3]. In standard relativity, the concept of time is completely different from what we experience: there is no observer-independent flow of time and there is no simple concept of present. Instead, the idea of eternalism (the block universe theory) appears to be the notion of time that best corresponds with standard relativity. These are significant changes to our understanding of the world, which we should challenge if we wish to make sure that they are really correct. We will therefore examine in this paper a theory of gravity which is based on the Lorentz-Poincaré interpretation. That theory interprets gravity as Archimedes' thrust exerted by a perfect fluid or "ether" on the matter particles — those being viewed as extended objects, more precisely as organized flows in that same fluid. This interpretation of gravitation has been discussed in detail [4]. It couples naturally with the Lorentz-Poincaré interpretation of special relativity. This leads to a relativistic theory of gravitation with a preferred reference frame, based on a unique scalar field [5]: hence the name "scalar ether theory" or SET. The other motivations for that theory have been discussed in detail recently ([6], §1). Even more recently, in order to formulate a consistent electrodynamics in the presence of gravitation for that theory, one had to postulate an interaction energy, and that energy turns out to be a possible candidate for dark matter [7]. The experimental check of such an alternative theory of gravitation involves of course many points, a good number of which have been already checked, see among others Refs. [5, 8, 9]; for a summary, see again Ref. [6], §1. (As noted there, this scalar theory differs from all known scalar theories.) In particular, the celestial mechanics has also been checked for this theory [8, 10], but this was for an earlier version of the theory ("v1", see Ref. [9] and references therein), which had to be modified to the current version ("v2", see Ref. [5]). The aim of the present research is to begin the check of the celestial mechanics of the "new" version, v2. That beginning consists essentially in assuming for simplicity that the mass centers of the N bodies move as test particles in the gravitational field of the other bodies. In this framework, the main task of the present work was to derive a tractable equation of motion for a test particle in SET.

In Section 2, we present the main equations of the theory. We note there that a general expression of the acceleration of a test particle obtained for v1 holds true for v2, and we show the spatial covariance of the equations. Section 3 derives a simple and exact expression of the acceleration in the most general case. In Sect. 4 we prove that, in the case where the gravitational field is produced by a uniformly moving massive body, the source of that field can be defined in the uniformly moving frame as a time-independent scalar field. We obtain then the explicit exact solution for the gravitational field, Eq. (50). That case is further specialized in Sect. 5 by assuming that the massive body has spherical symmetry. We provide there the expression of the acceleration both in the preferred frame and in the moving frame. Section 6 discusses the application to the effective calculation of an ephemeris. [1] We show that the approximation done in the present work to calculate the post-Newtonian (PN) *correction*, that each planet moves as a test particle in the field of the spherical Sun, plus the fact that comparison is made with an ephemeris based on GR, necessarily lead to the result found, that the velocity of the barycenter is unrealistically small. Therefore, the next steps should be (i) to derive fully consistent equations at the PN level, that take into account the self fields (as was previously done for v1 [10]), and, most importantly (ii) to adjust the theory on "direct" data, as little affected as possible by a reduction using GR.

[1] In order to test this theory in celestial mechanics, it is preferable to calculate ephemerides. In any case, one cannot use the parameterized PN formalism, since in that theory the test particles generally do not have a geodesic motion [5, 11].

2 Main equations of the theory

A) Metrics. The theory which is studied (SET) is a scalar theory with a preferred reference frame and two Lorentzian spacetime metrics: a flat one (Minkowski's metric) $\boldsymbol{\gamma}^0$ and a curved or "physical" one $\boldsymbol{\gamma}$, the latter being related to $\boldsymbol{\gamma}^0$ through a scalar field [5]. The preferred reference fluid [2] $\mathcal{E}$ assumed by the theory is an inertial frame for the flat metric $\boldsymbol{\gamma}^0$, i.e., there are spacetime coordinates x^μ which are both *adapted* to $\mathcal{E}$ [3] and Cartesian for $\boldsymbol{\gamma}^0$ — that is, $\gamma^0_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ $(\mu, \nu = 0, ..., 3)$ are the components of the standard matrix $\eta = (\eta_{\mu\nu}) := \mathrm{diag}(1, -1, -1, -1)$. The time $T := x^0/c$ is well defined up to a constant shift [4] and is a preferred time: the inertial time in the inertial frame $\mathcal{E}$. (Here, c is the velocity of light.) In such coordinates, we have also, by assumption:

$$\gamma_{0i} = 0 \quad (i = 1, 2, 3). \tag{1}$$

The latter equation ("synchronization condition") implies that the spatial metric $\boldsymbol{g}$ associated in the reference fluid $\mathcal{E}$ with the spacetime metric $\boldsymbol{\gamma}$ [12, 16, 17] is just the spatial part of the spacetime metric $\boldsymbol{\gamma}$, i.e. $g_{ij} = -\gamma_{ij}$ $(i, j = 1, 2, 3)$ in such coordinates [16]. The spatial metric associated in the reference fluid $\mathcal{E}$ with the flat metric $\boldsymbol{\gamma}^0$ is an *Euclidean* (i.e. flat and Riemannian) spatial metric $\boldsymbol{g}^0$ that is time-independent, i.e., $\partial g^0_{ij}/\partial x^0 = 0$ in any coordinates that are adapted to $\mathcal{E}$. The metric $\boldsymbol{g}$ is assumed to have a simple relation with $\boldsymbol{g}^0$:

$$\boldsymbol{g} = \beta^{-2}\boldsymbol{g}^0, \tag{2}$$

[2] A *reference fluid* is a 3-D congruence of reference world lines, each of which defines the trajectory of a point bound to that reference fluid [12]. This notion does not imply the presence of a real fluid. However, in the case of the preferred reference fluid $\mathcal{E}$, we do imagine (at a heuristic level) that it represents the averaged motion of some kind of fluid, the "micro-ether", see §3.3 in Ref. [4].

[3] Spacetime coordinates x^μ *adapted* to a given reference fluid $\mathcal{F}$ are ones for which the reference world lines have constant spatial coordinates x^i $(i = 1, 2, 3)$ [13, 14]. For a change from one set of coordinates adapted to $\mathcal{F}$ to another one (i.e., for an "internal transformation of coordinates" [12]), the change in the spatial coordinates has to be independent of the time coordinate, but the change in the time coordinate is arbitrary [12]. Hence we can speak of adapted spatial coordinates x^i as well. By a *reference frame,* we mean a reference fluid endowed with a given time coordinate map. For more detail on these notions and their development, see Ref. [15], and references therein.

[4] If x'^μ are other coordinates that verify those two conditions, we have $\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\eta_{\rho\sigma} = \eta_{\mu\nu}$ and $\frac{\partial x'^i}{\partial x^0} = 0$ $(i = 1, 2, 3)$. It follows easily that $\partial x'^0/\partial x^i = 0$ and $\partial x'^0/\partial x^0 = \pm 1$. Therefore, asking $\partial x'^0/\partial x^0 > 0$, the time coordinate is well defined up to a constant shift.

where

$$\beta := \sqrt{\gamma_{00}}. \tag{3}$$

This means that, in a gravitational field, the measuring rods are contracted and the periods of clocks are dilated, in the same ratio β (usually $\beta \leq 1$). The reason for this assumption in the framework of the hypothesis of a perfectly fluid "ether" is explained in detail in Ref. [4]. Thus, in Cartesian coordinates that are adapted to the reference fluid $\mathcal{E}$, we can write the line elements of the flat metric and the "physical" metric respectively as

$$(\mathrm{d}s^0)^2 := \gamma^0_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = (\mathrm{d}x^0)^2 - \mathrm{d}x^i\mathrm{d}x^i, \tag{4}$$

$$\mathrm{d}s^2 := \gamma_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \beta^2(\mathrm{d}x^0)^2 - \beta^{-2}\mathrm{d}x^i\mathrm{d}x^i. \tag{5}$$

B) Equation of motion of a test particle. In SET, motion is defined by an extension to curved spacetime of the special-relativistic form of Newton's second law: force = time-derivative of momentum, the latter involving either the velocity-dependent relativistic mass or the energy of the photon (divided by c^2) [11]. That extension involves an acceleration vector $\mathbf{g}$ of gravitation, defined as follows:

$$\mathbf{g} := -c^2 \, \frac{\mathrm{grad}_{\boldsymbol{g}}\, \beta}{\beta}, \tag{6}$$

where

$$\left(\mathrm{grad}_{\boldsymbol{g}} \beta\right)^i := g^{ij}\beta_{,j}, \tag{7}$$

$(g^{ij}) := (g_{ij})^{-1}$ being the inverse matrix of matrix (g_{ij}). The precise writing of the curved-spacetime Newton second law has been discussed in detail in Ref. [11] and has been summarized, for example, in Ref. [6]: Sect. 2, Point (**iii**). Here we will need only the "coordinate acceleration" which has been deduced from it in Ref. [18], Eq. (18) there:

$$\frac{\mathrm{d}u^i}{\mathrm{d}T} = \frac{1}{\beta}\left(\frac{\partial\beta}{\partial T} + 2\beta_{,j}u^j\right)u^i - \Gamma^i_{jk}u^j u^k - \frac{1}{2}g^{ij}\,\frac{\partial g_{jk}}{\partial T}u^k - \frac{c^2}{2}\,f(\nabla_0 f)^i, \tag{8}$$

with $u^i := \mathrm{d}x^i/\mathrm{d}T$, $f := \beta^2$, $\nabla_0 := \mathrm{grad}_{\boldsymbol{g}^0}$, and where the Γ^i_{jk} 's are the Christoffel symbols associated with the spatial metric $\boldsymbol{g}$. In Ref. [18], which belonged to a first version of SET ("v1"), the assumed relation between the two spatial metrics differed from Eq. (2) above. However, the derivation of Eq. (8) above depended of that relation only through the following re-expression of the space vector (6) (Eq. (2) in Ref. [18]):

$$\mathbf{g} = -\frac{c^2}{2}\nabla_0\, f. \tag{9}$$

Now, with the relation between the two spatial metrics assumed in the second version of SET ("v2") [4, 5], Eq. (2) above, we have for the inverse matrices (g^{ij}) and $(g^{0\,ij})$:

$$g^{ij} = \beta^2\, g^{0\,ij}, \tag{10}$$

so that the gravity acceleration (6) is:

$$g^i := -c^2\frac{g^{ij}\beta_{,j}}{\beta} = -c^2\frac{\beta^2\, g^{0\,ij}\beta_{,j}}{\beta} = -c^2\beta g^{0\,ij}\beta_{,j} = -\frac{c^2}{2}g^{0\,ij}f_{,j} := -\frac{c^2}{2}(\nabla_0\, f)^i. \tag{11}$$

Therefore, Eq. (9) remains valid for v2, and hence the same is true for Eq. (8). We emphasize that the equation of motion (8) is valid for a massive test particle as well as for a photon [18].

C) Equation for the scalar field. This is the flat-spacetime wave equation for the scalar field $\psi := -\mathrm{Log}\,\beta$ [5]:

$$\Box\psi := \psi_{,0,0} - \Delta_{\boldsymbol{g}^0}\,\psi = \frac{4\pi G}{c^2}\sigma. \tag{12}$$

Here $\Delta_{\boldsymbol{g}^0}$ is the usual Laplace operator, defined with the Euclidean space metric $\boldsymbol{g}^0$ (thus $\Delta_{\boldsymbol{g}^0}\,\psi = \psi_{,i,i}$ in Cartesian coordinates for $\boldsymbol{g}^0$, i.e. such that $g^0_{ij} = \delta_{ij}$). And σ is the energy component of the total energy-momentum tensor $\boldsymbol{T}$ of matter and non-gravitational fields in the reference frame $\mathcal{E}$ (i.e., the reference fluid $\mathcal{E}$ endowed with the preferred time coordinate $x^0 = cT$, with T the inertial time in $\mathcal{E}$):

$$\sigma := \left(T^{00}\right)_{\mathcal{E}}. \tag{13}$$

(We take $\boldsymbol{T}$ in mass units, i.e., it is $c^2\sigma$ which is truly a volume energy density.)

D) Covariance. A first point is that all equations written above, except for Eqs. (4), (5), and (8), are manifestly covariant under any purely spatial coordinate change

$$x'^0 = x^0, \qquad x'^i = \psi^i(x^1, x^2, x^3). \tag{14}$$

For instance, the fact that Eq. (6) is covariant under any change (14) results immediately from the facts that β [Eq. (3)] is obviously invariant under such a change and that then Eqs. (6)–(7) manifestly define a *spatial vector*, [5] i.e., after such a change the components g^i become

$$g'^i = \frac{\partial x'^i}{\partial x^j}\, g^j. \tag{15}$$

Similarly, γ_{0i} $(i = 1, 2, 3)$ are manifestly the components of a spatial vector, hence Eq. (1), if it is valid in some coordinate system, remains valid after any spatial change. Moreover, rewriting Eqs. (4) and (5) after a change (14) is easy: replace $\mathrm{d}x^i\mathrm{d}x^i$ with $g^0_{ij}\mathrm{d}x^i\mathrm{d}x^j$. The coordinates obtained after a change (14) are still adapted to the preferred reference frame $\mathcal{E}$, but of course the new spatial coordinates x'^i are generally not Cartesian for $\boldsymbol{g}^0$. As to the acceleration (8): it is not a spatial vector [18], but Eq. (8) becomes (manifestly) space-covariant if one puts the term $\Gamma^i_{jk}u^j u^k$ on the l.h.s., thus expressing the "absolute" derivative of the 3-vector $\mathbf{u}$ w.r.t. T (and based on the metric $\boldsymbol{g}$ of the time considered [11]), instead of its total derivative w.r.t. T — so that also Eq. (8) holds true after any purely spatial coordinate change (14). Thus, as one expects from a preferred-frame theory, all equations written above are spatially covariant, except for Eqs. (4) and (5) which are easily rewritten in spatially-covariant form.

In addition, the "synchronization condition" (1), as well as the definition of the gravity acceleration (6), are stable also by a change of the time coordinate having the special form

$$x'^0 = \varphi(x^0). \tag{16}$$

However, the relation (2) between the flat and the curved spatial metrics, as well as the field equation (12), are valid only if the time coordinate is $x^0 = cT$ — and in any spatial coordinates that are adapted to $\mathcal{E}$. Indeed β

[5] A "geometric" definition also exists for this: a spatial vector is a vector in the tangent space, at some point, to the space manifold associated with the reference fluid $\mathcal{E}$ (see Ref. [15], §4.2).

is not invariant under a change (16) of the time coordinate, in contrast with $\boldsymbol{g}$ and $\boldsymbol{g}^0$, hence Eq. (2) is not covariant under such a change. Hence neither is the rewriting (11) of the vector $\mathbf{g}$, nor the equation of motion in the form (8). The scalar field of the theory has therefore to be defined more precisely (in spatial coordinates adapted to $\mathcal{E}$) to be [5, 9]:

$$\tilde{\beta} := (\sqrt{\gamma_{00}})_{x^0=cT}\,. \tag{17}$$

In this paper, we take $x^0 = cT$ as the time coordinate, hence we may forget the distinction between β and $\tilde{\beta}$.

3 Exact equation of motion of a test particle

The equation of motion (8) does not fully take into account the explicit form (2) of the spatial metric $\boldsymbol{g}$, e.g. it is valid also for the first version of SET, for which it was first derived. With (2), we have in Cartesian coordinates for the Euclidean metric $\boldsymbol{g}^0$: $g_{ij} = \phi\delta_{ij}$ with $\phi = \beta^{-2}$, hence the Christoffel symbols of $\boldsymbol{g}$ are given by:

$$\begin{aligned}
\Gamma^i_{jk} &:= \frac{1}{2}g^{il}\left(g_{lj,k} + g_{lk,j} - g_{jk,l}\right) \\
&= \frac{\phi^{-1}}{2}\delta^{il}\left(\phi_{,k}\delta_{lj} + \phi_{,j}\delta_{lk} - \phi_{,l}\delta_{jk}\right) \\
&= \frac{\phi^{-1}}{2}\left(\phi_{,k}\delta^i_j + \phi_{,j}\delta^i_k - \phi_{,i}\delta_{jk}\right) \\
&= \frac{-1}{\beta}\left(\beta_{,k}\delta^i_j + \beta_{,j}\delta^i_k - \beta_{,i}\delta_{jk}\right).
\end{aligned} \tag{18}$$

We get also from (2):

$$\frac{1}{2}g^{ij}g_{jk,T} = \frac{-\beta_{,T}}{\beta}\,\delta^i_k. \tag{19}$$

We thus obtain from (8):

$$\frac{\mathrm{d}u^i}{\mathrm{d}T} = \frac{1}{\beta}\left(\beta_{,T} + 2\beta_{,j}\,u^j\right)u^i + \frac{1}{\beta}\left(\beta_{,k}\,u^i\,u^k + \beta_{,j}\,u^j\,u^i - \beta_{,i}\,u^j\,u^j\right) + \frac{\beta_{,T}}{\beta}\,u^i - \frac{c^2}{2}f(\nabla_0\,f)^i. \tag{20}$$

In Cartesian coordinates for the Euclidean metric $\boldsymbol{g}^0$, the last term is

$$-\frac{c^2}{2} f(\nabla_0\, f)^i = -c^2 \beta^3 \beta_{,i}. \tag{21}$$

Hence we can rewrite (20) as

$$\frac{\mathrm{d}u^i}{\mathrm{d}T} = \frac{1}{\beta}\left[\left(2\beta_{,T} + 4\beta_{,j}\, u^j\right) u^i - \beta_{,i}\, u^j\, u^j\right] - c^2\beta^3\beta_{,i}. \tag{22}$$

We still reexpress this in terms of the field $\psi := -\mathrm{Log}\,\beta$ that enters the field equation (12). We have

$$\frac{\beta_{,\mu}}{\beta} = \left(\mathrm{Log}\,\beta\right)_{,\mu} = -\psi_{,\mu} \tag{23}$$

and

$$\beta^3\beta_{,i} = \beta^4 \frac{\beta_{,i}}{\beta} = -e^{-4\psi}\psi_{,i} \tag{24}$$

so that Eq. (22) rewrites as

$$\frac{\mathrm{d}u^i}{\mathrm{d}T} = -2\psi_{,T}u^i - 4\psi_{,j}\, u^j\, u^i + \psi_{,i}\, u^j\, u^j + c^2 e^{-4\psi}\psi_{,i}. \tag{25}$$

4 Case of a uniformly moving massive body

Consider a massive body, say B, which is in a translation, at a uniform and constant velocity $\mathbf{V}$, with respect to the preferred reference frame $\mathcal{E}$. This means that, at any time T, the spatial velocity vector $\mathbf{u}$ of any point bound with B is the same spatial vector $\mathbf{V}$. [6] Hence the spatial positions of that point at time $T = 0$ and at time T fulfil

$$\mathbf{x}(T) - \mathbf{x}(T=0) = \tilde{\mathbf{V}}T. \tag{26}$$

[6] The velocity of a given point bound with B is in general a time-dependent spatial vector in the reference frame $\mathcal{E}$, having components $u^i(T) = \mathrm{d}x^i/\mathrm{d}T$ in any spatial coordinates x^i adapted to $\mathcal{E}$. The *uniformity*, at any given time T, of the 3-vector $\mathbf{V}$, refers to the connection associated with the Euclidean metric $\boldsymbol{g}^0$: it means that $\mathbf{V}$ is parallely transported along any spatial curve $x^i = x^i(\xi)$. It thus means that the components V^i stay unchanged ($\mathrm{d}V^i/\mathrm{d}\xi = 0$) along any curve in any *Cartesian* spatial coordinates adapted to $\mathcal{E}$ (Note 3), in other words V^i does not depend on the spatial position. The *constancy* of $\mathbf{V}$ means that this spatially uniform vector field actually does not depend on T, $\mathrm{d}V^i/\mathrm{d}T = 0$. Thus V^i is a true constant in any Cartesian spatial coordinates adapted to $\mathcal{E}$.

Here $\mathbf{x}(T) := (x^i(T)) \in \mathbb{R}^3$, the x^i 's being any Cartesian spatial coordinates adapted to $\mathcal{E}$, and accordingly $\tilde{\mathbf{V}} := (V^i) \in \mathbb{R}^3$. [7] The energy density relative to $\mathcal{E}$, Eq. (13), follows that translation and verifies hence:

$$\sigma(T, \mathbf{x} + \mathbf{V}T) = \sigma(T = 0, \mathbf{x}). \tag{27}$$

This relation remains true in a more general situation, in which the body has, in addition to its uniform translation, a stationary rotation with an axis that follows the translation at $\mathbf{V}$, and around which the energy distribution σ is rotationally symmetric. Consider now the global reference fluid, say $\mathcal{E}_{\mathbf{V}}$, that follows (only) the uniform translation, at velocity $\mathbf{V}$, of body B. That is: at any time T, the spatial velocity vector $\mathbf{u}$ of any point bound with $\mathcal{E}_{\mathbf{V}}$ is the same spatial vector $\mathbf{V}$ — but now the initial position $\mathbf{x}(T = 0)$ in Eq. (26) can be any vector $\mathbf{x}_0 \in \mathbb{R}^3$. The reference fluid $\mathcal{E}_{\mathbf{V}}$ is also an inertial frame for the flat metric $\boldsymbol{\gamma}^0$, as is $\mathcal{E}$. The reference fluid $\mathcal{E}_{\mathbf{V}}$, endowed with its own inertial time T', will henceforth be called "the moving frame". We can define the field σ, as it is "seen" in the moving frame, as follows:

$$\sigma'(\mathbf{X}') := \sigma(\mathbf{X}(\mathbf{X}')) \qquad (\sigma(\mathbf{X}) := (T^{00})_{\mathcal{E}}(\mathbf{X})). \tag{28}$$

Here, $\mathbf{X}' := (x'^\mu)$ are spacetime coordinates adapted to the moving reference fluid $\mathcal{E}_{\mathbf{V}}$, and with $x'^0 = cT'$; whereas $\mathbf{X} := (x^\mu)$ are spacetime coordinates adapted to the preferred frame $\mathcal{E}$, and with $x^0 = cT$. Of course, σ' is in general *not* the T^{00} component in the new coordinates, i.e., $\sigma'(\mathbf{X}') \neq T'^{00}(\mathbf{X}')$ in general, since

$$T'^{00}(\mathbf{X}') = \frac{\partial x'^0}{\partial x^\mu}\frac{\partial x'^0}{\partial x^\nu} T^{\mu\nu}(\mathbf{X}) \neq T^{00}(\mathbf{X}) = \sigma(\mathbf{X}), \tag{29}$$

and since the latter is by the definition (28) equal to $\sigma'(\mathbf{X}')$.

Proposition. *If the relation (27) is true, then the field σ' defined by (28) does not depend on the inertial time T' in $\mathcal{E}_{\mathbf{V}}$:*

$$\sigma' = \sigma'(x'^1, x'^2, x'^3) = \sigma'(\mathbf{x}'). \tag{30}$$

[7] Thus $\mathbf{x}$ and $\tilde{\mathbf{V}}$ are "coordinate 3-vectors". They are not spatial vectors (see after Eq. (15)). This is because to define the spatial position as a spatial vector one needs to choose an origin point. When the coordinate system is changed, $\mathbf{x}$ and $\tilde{\mathbf{V}}$ change to $\mathbf{x}' := (x'^i)$ and $\tilde{\mathbf{V}}' := (V'^i)$, but the spatial vector $\mathbf{V}$ that has the V^i 's as components in the first coordinate system and the V'^i 's in the second one, is the same vector. Hereafter, for the simplicity of notation, $\mathbf{V}$ will denote both the spatial vector and the "coordinate vector".

Proof. Note first that the energy density (13) involved in the relation (27) is invariant under any purely spatial coordinate change (14). Hence, if (27) is true for one set of spatial coordinates x^i which are adapted to $\mathcal{E}$ and are Cartesian for the Euclidean metric $\boldsymbol{g}^0$, then it remains true if we change the spatial coordinates for ones with the same property. Also, if (30) is true with one set of coordinates x'^i (whether they are Cartesian for some Euclidean metric or not), it holds true after a purely spatial coordinate change (14) applied to the x'^i 's. So we can choose the spatial coordinates x^i (adapted to $\mathcal{E}$ and Cartesian for $\boldsymbol{g}^0$) and the spatial coordinates x'^i (adapted to $\mathcal{E}_{\mathbf{V}}$) as we wish. Starting from coordinates $(x^\mu) = (cT, x, y, z)$, which are adapted to $\mathcal{E}$ and with x, y, z being Cartesian for $\boldsymbol{g}^0$, we obtain coordinates adapted to the moving frame by doing a Lorentz transformation, which can be made special through the choice of the axes:

$$F: \quad T' = \gamma\left(T - \frac{Vx}{c^2}\right), \quad x' = \gamma(x - VT), \quad y' = y, \quad z' = z \tag{31}$$

(here $\gamma = (1-(V^2/c^2))^{-1/2}$ is the Lorentz factor). The inverse transformation is

$$F^{-1}: \quad T = \gamma\left(T' + \frac{Vx'}{c^2}\right), \quad x = \gamma(x' + VT'), \quad y = y', \quad z = z'. \tag{32}$$

Let us evaluate $\sigma'(T', \mathbf{x}') - \sigma'(T' = 0, \mathbf{x}')$. To apply (28) we define

$$(T, \mathbf{x}) := F^{-1}(T', \mathbf{x}') \qquad \text{and} \quad (T_0, \mathbf{x}_0) := F^{-1}(T' = 0, \mathbf{x}'). \tag{33}$$

To obtain $(T_0, \mathbf{x}_0)$ we first apply (32) with $T' = 0$ and get

$$T_0 = \gamma\frac{Vx'}{c^2}, \quad x_0 = \gamma x', \quad y_0 = y', \quad z_0 = z'. \tag{34}$$

Then, since from (33) we have $(T', \mathbf{x}') = F(T, \mathbf{x})$, we enter precisely (31) into the latter equation to get

$$T_0 = \frac{\gamma^2 V(x - VT)}{c^2}, \quad x_0 = \gamma^2(x - VT), \quad y_0 = y, \quad z_0 = z. \tag{35}$$

Now we want to use (27). We rewrite it as

$$\sigma(T_1, \mathbf{x} + \mathbf{V}T_1) = \sigma(T = 0, \mathbf{x}) = \sigma(T_2, \mathbf{x} + \mathbf{V}T_2) \tag{36}$$

(for any given $\mathbf{x}$ and any two times T_1 and T_2), from which we see that

$$\sigma(T_1, \mathbf{x}_1) = \sigma(T_2, \mathbf{x}_2) \quad \text{if} \quad \mathbf{x}_2 - \mathbf{x}_1 = \mathbf{V}(T_2 - T_1). \tag{37}$$

(Note from (35) that y and z are constant in our current manipulations, corresponding with the fact that $\mathbf{V} = (V, 0, 0)$.) We apply (37) with $(T_1, \mathbf{x}_1) := (T_0, \mathbf{x}_0)$ defined in Eq. $(33)_2$ and computed in Eq. (35). So (37) allows us to write:

$$\sigma(T_0, \mathbf{x}_0) = \sigma(T_0, x_0, y_0, z_0) = \sigma(T_0, x_0, y, z) = \sigma(T_2, x, y, z) \tag{38}$$

provided that

$$V(T_2 - T_0) = x - x_0. \tag{39}$$

From this and (35) we compute the corresponding time T_2 as

$$T_2 = T_0 + \frac{x - x_0}{V} = \frac{\gamma^2 V(x - VT)}{c^2} + \frac{x - \gamma^2(x - VT)}{V}, \tag{40}$$

that is

$$T_2 = x\left(\frac{\gamma^2 V}{c^2} - \frac{\gamma^2 - 1}{V}\right) + T\left(\gamma^2 - \frac{\gamma^2 V^2}{c^2}\right) = T. \tag{41}$$

This and (38) mean that

$$\sigma(T_0, \mathbf{x}_0) = \sigma(T, \mathbf{x}). \tag{42}$$

But by the definitions (28) and (33), we have precisely

$$\sigma(T_0, \mathbf{x}_0) = \sigma'(0, \mathbf{x}') \qquad \text{and} \quad \sigma(T, \mathbf{x}) = \sigma'(T', \mathbf{x}'). \tag{43}$$

We thus obtain that, for any value T' of the time of the moving frame and for any position $\mathbf{x}'$ in this frame:

$$\sigma'(0, \mathbf{x}') = \sigma'(T', \mathbf{x}'). \tag{44}$$

This proves the Proposition. □

Remark: If instead of the Lorentz transformation (31) the coordinate transformation is the Galileo transformation:

$$(T', \mathbf{x}') = F(T, \mathbf{x}) := (T, \mathbf{x} - \mathbf{V}T), \tag{45}$$

then the same Proposition is true also, its proof being then much simpler. On the other hand, if the coordinate transformation F is fully general (and in particular non-linear), then one cannot deduce anything like this. What one can write then is:

$$\begin{aligned}\sigma'(T',\mathbf{x}') &:= \sigma(T,\mathbf{x}) \text{ with } (T,\mathbf{x}) := F^{-1}(T',\mathbf{x}') \\ &= \sigma(0,\mathbf{x}-\mathbf{V}T) \\ &= \sigma'(T'_0,\mathbf{x}'_0) \text{ with } (T'_0,\mathbf{x}'_0) := F(0,\mathbf{x}-\mathbf{V}T), \end{aligned} \tag{46}$$

and this cannot be transformed further to obtain (30).

In the same way as in Eq. (28), we define

$$\psi'(\mathbf{X}') := \psi(\mathbf{X}(\mathbf{X}')). \tag{47}$$

Since the flat wave operator is (in particular) Lorentz-invariant, it follows from (12), (28) and (47) that we have when the coordinates x'^{μ} (adapted to $\mathcal{E}_{\mathbf{V}}$ and such that $x'^0 = cT'$) are moreover Cartesian for the Minkowski metric $\boldsymbol{\gamma}^0$:

$$\Box\psi' = \frac{\partial^2\psi'}{\partial(x'^0)^2} - \frac{\partial^2\psi'}{\partial x'^i\partial x'^i} = \kappa\sigma', \qquad \kappa := \frac{4\pi G}{c^2}. \tag{48}$$

The relevant solution for an (assumed) isolated massive body B is the pure retarded potential (i.e. without the addition of a solution of the homogeneous wave equation), because it corresponds to the situation without an external field [16, 19]. Thus

$$\psi'(T',\mathbf{x}') = \frac{\kappa}{4\pi}\int_{\mathrm{B}} \sigma'\left(T' - \frac{|\mathbf{x}'-\mathbf{y}'|}{c}, \mathbf{y}'\right)\frac{\mathrm{d}^3\mathbf{y}'}{|\mathbf{x}'-\mathbf{y}'|}. \tag{49}$$

However, due to the time-independence of σ' (30), the retardation has no effect and we get a stationary field:

$$\psi' = \psi'(\mathbf{x}') = \frac{G}{c^2}\int_{\mathrm{B}} \frac{\sigma'(\mathbf{y}')}{|\mathbf{x}'-\mathbf{y}'|}\,\mathrm{d}^3\mathbf{y}'. \tag{50}$$

The exact solution (50) for the gravitational potential ψ is thus just like the Newtonian potential, but remind that it has to be transformed to the preferred frame $\mathcal{E}$ by a Lorentz transformation. In order to use (50) in the equation of motion (25), we have to express the derivatives w.r.t. the "fixed" coordinates x^μ in terms of derivatives w.r.t. the "moving" coordinates x'^μ. This will be done in the next section in the case of spherical symmetry. In practice the velocity vector $\mathbf{V}$ is an unknown, more precisely it is a "solved-for parameter" in the optimization software [10], see Sect. 6. As a consequence, we cannot use a special Lorentz transformation, instead we use the general version of the Lorentz transformation (e.g. Weinberg [20], §2.1):

$$T' = g(T, \mathbf{x}) := \gamma \left(T - \frac{\mathbf{V}.\mathbf{x}}{c^2} \right), \tag{51}$$

$$\mathbf{x}' = \mathbf{f}(T, \mathbf{x}) := \mathbf{x} + \frac{\gamma - 1}{V^2} \, (\mathbf{V}.\mathbf{x}) \, \mathbf{V} - \gamma T \mathbf{V}. \tag{52}$$

From this, we find easily using the fact that $\partial \psi' / \partial T' = 0$ (Eq. (50)):

$$(\nabla_{\mathbf{x}} \psi)(T, \mathbf{x}) = (\nabla_{\mathbf{x}'} \psi')(\mathbf{f}(T, \mathbf{x})) + (\gamma - 1) \, (\mathbf{V}.(\nabla_{\mathbf{x}'} \psi')(\mathbf{f}(T, \mathbf{x}))) \, \frac{\mathbf{V}}{V^2} \tag{53}$$

$$= \nabla_{\mathbf{x}'} \psi' + (\mathbf{V}.\nabla_{\mathbf{x}'} \psi') \, \frac{\mathbf{V}}{2c^2} + O(c^{-4}) \tag{54}$$

(all equations without an $O(c^{-n})$ remainder are exact), and also

$$\frac{\partial \psi}{\partial T}(T, \mathbf{x}) = \frac{\partial \psi'}{\partial x'^j}(\mathbf{f}(T, \mathbf{x})) \frac{\partial f^j}{\partial T}(T, \mathbf{x}) = -\gamma V^j \frac{\partial \psi'}{\partial x'^j} = -\gamma \mathbf{V}.\nabla_{\mathbf{x}'} \psi'. \tag{55}$$

5 Case of a spherical uniformly moving massive body

Let us assume that the source σ' in Eq. (50) is spherically symmetric, i.e.

$$\sigma'(\mathbf{x}') = \sigma'(r'), \qquad r' := |\mathbf{x}' - \mathbf{x}'_b|, \tag{56}$$

where $\mathbf{x}'_b$ is the (fixed) position, in the moving frame $\mathcal{E}_\mathbf{V}$, of the center of spherical symmetry (the mass center of body B) and $|\mathbf{x}'| := (\boldsymbol{g}'^0(\mathbf{x}', \mathbf{x}'))^{1/2} =$

$x'^i\, x'^i$, with $\boldsymbol{g}'^0$ the Euclidean spatial metric associated with the Minkowski metric $\boldsymbol{\gamma}^0$ in the moving frame $\mathcal{E}_{\mathbf{V}}$. [8] Then we get:

$$\psi'(r') = \frac{G\, M}{c^2\, r'}, \tag{57}$$

and

$$(\nabla_{\mathbf{x}'}\psi')(\mathbf{x}') = -\frac{GM}{c^2}\frac{\mathbf{x}' - \mathbf{x}'_b}{r'^3}, \tag{58}$$

with

$$M := \int_{\mathrm{B}} \sigma'(r')\, \mathrm{d}^3\mathbf{x}'. \tag{59}$$

(Of course, Eqs. (57) and (58) are valid only outside the massive body B, that is for $r' > R$, where R is the radius of B, i.e., $\sigma'(r') = 0$ for $r' > R$. Note that the assumption that the energy density be spherically symmetric is indeed more correct in the frame that moves with the body, for there is then no Lorentz contraction to account for.) We note $\mathbf{x}_a(T)$ and $\mathbf{x}_b(T)$ the positions in the preferred frame of a test particle (which could be the mass center of a planet) and of the center of body B (which could be the Sun), and

$$\mathbf{r}_{ab}(T) := \mathbf{x}_a(T) - \mathbf{x}_b(T). \tag{60}$$

The positions in the moving frame are related to $\mathbf{x}_a(T)$ and $\mathbf{x}_b(T)$ by the Lorentz transformation (51)–(52):

$$\mathbf{x}'_a(T) := \mathbf{f}(T, \mathbf{x}_a(T)), \qquad \mathbf{f}(T, \mathbf{x}_b(T)) = \mathbf{x}'_b = \text{Constant}. \tag{61}$$

Setting

$$\mathbf{r}'_{ab}(T) := \mathbf{x}'_a(T) - \mathbf{x}'_b, \qquad R'_{ab}(T) := \left|\mathbf{r}'_{ab}(T)\right|, \tag{62}$$

we have from (58):

$$(\nabla_{\mathbf{x}'}\psi')(\mathbf{x}'_a(T)) = \frac{GM}{c^2}\mathbf{h}', \qquad \mathbf{h}'(T) := -\frac{\mathbf{r}'_{ab}(T)}{R'^3_{ab}(T)}. \tag{63}$$

We can then use (53) and (55) to rewrite the acceleration (25) more explicitly as

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}T} = \frac{GM}{c^2}\left[-2h_V\mathbf{u} - 4(\mathbf{h}.\mathbf{u})\mathbf{u} + \mathbf{u}^2\mathbf{h} + c^2\beta^4\mathbf{h}\right], \tag{64}$$

[8] Since the Lorentz transformation (51)–(52) transforms the Cartesian coordinates x^μ for $\boldsymbol{\gamma}^0$ to Cartesian coordinates x'^μ for $\boldsymbol{\gamma}^0$, it follows that the new spatial coordinates x'^i are Cartesian for the Euclidean spatial metric $\boldsymbol{g}'^0$. Simply: $(\mathrm{d}s^0)^2 = (\mathrm{d}x^0)^2 - \mathrm{d}x^i\,\mathrm{d}x^i = (\mathrm{d}x'^0)^2 - \mathrm{d}x'^i\,\mathrm{d}x'^i = (\mathrm{d}x'^0)^2 - g'^0_{ij}\,\mathrm{d}x'^i\,\mathrm{d}x'^j$.

in which, from (53):

$$\begin{aligned}\mathbf{h} &:= \frac{c^2}{GM}(\nabla_{\mathbf{x}}\psi)(T,\mathbf{x}_a(T)) = \mathbf{h}'(T) + (\gamma-1)\,\frac{\mathbf{V}.\mathbf{h}'}{V^2}\mathbf{V} \\ &= \mathbf{h}' + \frac{\mathbf{V}.\mathbf{h}'}{2c^2}\mathbf{V} + O(c^{-4}),\end{aligned} \tag{65}$$

and from (55):

$$h_V := -\gamma\mathbf{h}'.\mathbf{V}, \tag{66}$$

with, moreover (to be calculated at $\mathbf{x} = \mathbf{x}_a(T)$):

$$\beta = \beta(T,\mathbf{x}) = e^{-\psi(T,\mathbf{x})}, \tag{67}$$

where, according to Eq. (57),

$$\psi(T,\mathbf{x}) = \psi'(T',\mathbf{x}') = \frac{GM}{c^2 r'}. \tag{68}$$

We want to compute the (first) post-Newtonian (PN) approximation, that includes the $O(c^{-2})$ corrections to the Newtonian acceleration. The latter is order zero w.r.t. c^{-2} and comes from the $\frac{GM}{c^2}c^2\beta^4\mathbf{h} = GM\beta^4\mathbf{h}$ term in the acceleration (64). Indeed we compute from (67) and (68):

$$\beta^4 = 1 - 4\frac{GM}{c^2\,r'} + O\left(c^{-4}\right). \tag{69}$$

The 1 on the r.h.s. of (69) gives the term $GM\mathbf{h}$ in the acceleration (64); this contains the Newtonian acceleration $GM\mathbf{h}'$, see Eq. (65).

The acceleration (64) can be reexpressed in the moving frame $\mathcal{E}_{\mathbf{V}}$, by using the Lorentz transformations of the velocity:

$$\mathbf{u}' = \frac{1}{1-\mathbf{u}.\mathbf{V}/c^2}\left[\left(\frac{\mathbf{u}.\mathbf{V}(1-\gamma^{-1})}{\mathbf{V}^2} - 1\right)\mathbf{V} + \gamma^{-1}\mathbf{u}\right] \tag{70}$$

and the acceleration:

$$\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}T'} = \frac{\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}T}}{\gamma^2(1-\frac{\mathbf{u}.\mathbf{V}}{c^2})^2} - \frac{(\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}T}.\mathbf{V})\mathbf{V}(\gamma-\mathbf{1})}{V^2\gamma^3(1-\frac{\mathbf{u}.\mathbf{V}}{c^2})^3} + \frac{(\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}T}.\mathbf{V})\mathbf{u}}{c^2\gamma^2(1-\frac{\mathbf{u}.\mathbf{V}}{c^2})^3}. \tag{71}$$

Neglecting all terms of order c^{-4} or higher, and setting $\mathbf{r}' := \mathbf{r}'_{ab}$ and $r' := R'_{ab}$, we obtain after a somewhat tedious calculation:

$$\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}T'} = -\frac{GM}{r'^3}\left(\mathbf{r}' + \frac{\mathbf{A}}{c^2}\right) + O(c^{-4}), \tag{72}$$

with

$$\mathbf{A} = -\left(\mathbf{r}'.\mathbf{V} + 4\mathbf{r}'.\mathbf{u}'\right)\left(\mathbf{V} + \mathbf{u}'\right) + \left[\mathbf{u}'^2 + 4(\mathbf{u}'.\mathbf{V}) + 2V^2\right]\mathbf{r}' - 4\frac{GM}{r'}\mathbf{r}'. \tag{73}$$

Thus, that part of the acceleration which is independent of $\mathbf{V}$ is:

$$\left(\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}T'}\right)_{\mathbf{V}=\mathbf{0}} = -\frac{GM}{r'^3}\left\{\mathbf{r}' + \frac{1}{c^2}\left[\left(\mathbf{u}'^2 - 4\frac{GM}{r'}\right)\mathbf{r}' - 4\left(\mathbf{r}'.\mathbf{u}'\right)\mathbf{u}'\right]\right\} + O\left(c^{-4}\right). \tag{74}$$

This is exactly the PN acceleration of a test particle in an SSS (static spherically symmetric) field as found in GR from the spatially-isotropic SSS solution of GR. See e.g. Weinberg [20], Eqs. (9.5.3) and (9.5.14), or Ref. [5], Eq. (89). Equivalently, Eq. (74) is exactly the SSS case of the PN acceleration of a test particle as found in GR when one starts from the so-called "standard PN metric". (The latter is spatially isotropic, see e.g. Weinberg [20], Eq. (9.1.60).) In fact, Eq. (74) can be easily checked directly by doing $\mathbf{V} = \mathbf{0}$ in the acceleration in the preferred frame, Eq. (64), since we have then

$$\mathbf{u} = \mathbf{u}', \quad h_V = 0, \quad \mathbf{h} = \mathbf{h}' = -\mathbf{r}'/r'^3, \tag{75}$$

and since we can use Eq. (69). The other part of the acceleration is:

$$\left(\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}T'}\right) - \left(\frac{\mathrm{d}\mathbf{u}'}{\mathrm{d}T'}\right)_{\mathbf{V}=\mathbf{0}} = \frac{GM}{c^2\,r'^3}\left\{\left[\mathbf{r}'.\left(\mathbf{V} + 4\mathbf{u}'\right)\right]\mathbf{V} + \left(\mathbf{r}'.\mathbf{V}\right)\mathbf{u}' - \left(4\mathbf{u}'.\mathbf{V} + 2V^2\right)\mathbf{r}'\right\} + O\left(c^{-4}\right). \tag{76}$$

Up to $O\left(c^{-4}\right)$, we can absorb the last term into the Newtonian acceleration through a redefinition of the active mass M to

$$M' := M\left[1 + 2(V^2/c^2)\right]. \tag{77}$$

6 Implementation in a software for ephemeris calculation

A) Principle, equations of motion. The equation of motion (64) has been implemented in a software for ephemeris calculation with parameter optimization. The main program loops on the numerical integration of the equations of motion for the major bodies of the solar system in order to minimize the least-squares residual [21]. The solved-for parameters of the optimization are: the initial conditions (position and velocity) for the N bodies, their masses, and (for SET only) the "absolute" velocity of their barycenter. That software had been built for v1, and it had been indeed used for v1 [10], after having been tested (i) with the Newtonian equations of motion (for spherical attracting bodies) [21] and (ii) with the Newtonian equations of motion modified to include, for the planets, the PN correction that comes from the Sun considered as a spherical body [22]. Here, besides using the same ODE integration scheme and the same optimization algorithm as in Refs. [21, 10, 22], we also use the same approximation as in the latter work [22] for the equations of motion. That is, in view of (64), we compute the acceleration, in the reference frame $\mathcal{E}$, of the planet with number a ($a = 1, ..., N-1$), as:

$$\frac{\mathrm{d}\mathbf{u}_a}{\mathrm{d}T} = \sum_{\substack{d=1 \\ d\neq a}}^{N-1} -\frac{G\,M_d\;(\mathbf{x}_a - \mathbf{x}_d)}{|\mathbf{x}_a - \mathbf{x}_d|^3} + \frac{GM_N}{c^2}\left[-2h_V\mathbf{u}_a - 4(\mathbf{h}.\mathbf{u}_a)\mathbf{u}_a + \mathbf{u}_a^2\mathbf{h} + c^2\,\beta^4\mathbf{h}\right]. \tag{78}$$

The last body, with number N, is the Sun. Its acceleration is computed as

$$\frac{\mathrm{d}\mathbf{u}_N}{\mathrm{d}T} = \sum_{d=1}^{N-1} -\frac{G\,M_d\;(\mathbf{x}_N - \mathbf{x}_d)}{|\mathbf{x}_N - \mathbf{x}_d|^3}. \tag{79}$$

Thus, in this model we take into account the Newtonian attractions of the planets on the Sun, hence the velocity $\mathbf{u}_N$ of the Sun w.r.t. $\mathcal{E}$ is not exactly a constant. To compute the contribution of the Sun to the acceleration of body a ($a < N$), i.e. the term with the square bracket on the r.h.s. of (78), we substitute $\mathbf{u}_N$ for $\mathbf{V}$ into the expressions (65) and (66). By doing so, essentially, we neglect the effect of the very small (and variable) acceleration of the Sun on the gravitational field that it produces. This is in addition to neglecting the departure from spherical symmetry of the gravitational fields of the Sun and the planets, to neglecting the PN corrections on the motion of the Sun that are due to the planets, and, perhaps most importantly [5, 10], to considering that the mass centers of the N bodies move as test particles in the gravitational field of the other bodies, i.e., essentially, to neglecting the effect of the self fields, which depend on the structure (density profile, self-rotation, etc.) [10].

B) Different frames. The data are referred to the heliocentric reference frame, say $\mathcal{H}$, whereas the equations of motion (78)–(79) are written in the preferred reference frame $\mathcal{E}$. To transform the initial conditions for the planets' positions and velocities from $\mathcal{H}$ to $\mathcal{E}$, we first transform them from $\mathcal{H}$ to the barycentric frame $\mathcal{B}$, by using the following coordinate transformation:

$$\mathbf{x}' = \mathbf{x}_h - \mathbf{x}_{B\,h}, \tag{80}$$

where $\mathbf{x}_h$ is the set of heliocentric spatial coordinates and

$$\mathbf{x}_{B\,h} = \left(\sum_{a=1}^{N-1} M_a \mathbf{x}_{a\,h}\right) / \sum_{a=1}^{N} M_a \tag{81}$$

is the heliocentric position of the barycenter (accounting for $\mathbf{x}_{N\,h} = \mathbf{0}$). The two reference frames $\mathcal{H}$ and $\mathcal{B}$ are endowed with the same time coordinate cT'. Accordingly, the corresponding heliocentric and barycentric velocities exchange by:

$$\mathbf{u}'_a := \frac{\mathrm{d}\mathbf{x}'_a}{\mathrm{d}T'} = \mathbf{u}_{a\,h} - \mathbf{u}_{B\,h} \quad (a = 1, ..., N), \qquad \mathbf{u}_{a\,h} := \frac{\mathrm{d}\mathbf{x}_{a\,h}}{\mathrm{d}T'}. \tag{82}$$

Then, assuming that the velocity of the barycenter w.r.t. the reference frame $\mathcal{E}$ is the vector $\mathbf{V}$, we transform the initial conditions from the uniformly moving frame $\mathcal{E}_\mathbf{V}$ to $\mathcal{E}$ by using the Lorentz transformation (51)–(52), or rather its obvious inverse. After having integrated the equations of motion (78)–(79) we do the reverse transformations. While going from $\mathcal{E}$ to $\mathcal{E}_\mathbf{V}$ or conversely, we have to correct the positions and velocities from "simultaneity gaps" (the fact that, e.g., the values T'_a of the time in $\mathcal{E}_\mathbf{V}$ which correspond to the events $(T, \mathbf{x}_a)$, that are simultaneous in $\mathcal{E}$, depend on a through Eq. (51)); we do that by using first-order Taylor expansions.

C) Variation of $\mathbf{V}$. The velocity $\mathbf{V}$ of the barycenter w.r.t. the reference frame $\mathcal{E}$:

$$\mathbf{V} := \frac{\mathrm{d}\mathbf{x}_B}{\mathrm{d}T} := \frac{\mathrm{d}}{\mathrm{d}T}\left(\sum_{a=1}^{N} \frac{M_a}{M_\mathrm{tot}}\,\mathbf{x}_a\right) = \sum_{a=1}^{N} \frac{M_a}{M_\mathrm{tot}}\,\mathbf{u}_a, \qquad M_\mathrm{tot} := \sum_{a=1}^{N} M_a, \tag{83}$$

is not exactly a constant in this model, since from Eqs. (78)–(79) we have

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}T} = \sum_{a=1}^{N} \frac{M_a}{M_\mathrm{tot}} \frac{\mathrm{d}\mathbf{u}_a}{\mathrm{d}T} = \sum_{a=1}^{N} \frac{M_a}{M_\mathrm{tot}} \left(\frac{\mathrm{d}\mathbf{u}_a}{\mathrm{d}T}\right)_\mathrm{N} + \sum_{a=1}^{N-1} \frac{M_a}{M_\mathrm{tot}} \left(\frac{\mathrm{d}\mathbf{u}_a}{\mathrm{d}T}\right)_\mathrm{PNc}, \tag{84}$$

where the index N indicates the Newtonian acceleration, given e.g. by Eq. (79) for the Sun, and where the index PNc means the PN *correction* to the acceleration of a planet. The latter correction is given for this model by the term with the square brackets in Eq. (78), minus the Newtonian acceleration due to the Sun. (See around Eq. (69).) The first sum in the rightmost side of Eq. (84) is like the sum of the Newtonian interaction forces in a system of point particles (divided by M_{tot}), and is hence zero by the actio-reactio principle. (This is easily checked directly.) We are thus left with

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}T} = \sum_{a=1}^{N-1} \frac{M_a}{M_{\mathrm{tot}}} \left(\frac{\mathrm{d}\mathbf{u}_a}{\mathrm{d}T}\right)_{\mathrm{PNc}} \neq \mathbf{0}. \tag{85}$$

In the present work (though not in the older work on v1 [10]), we have taken this into account. Updating $\mathbf{V}$ is done by adding it to the unknowns in the ODE solver, with Eq. (85) as the corresponding ODE, and with $\mathbf{V}(T_0)$ as the initial data. (It is now $\mathbf{V}_0 = \mathbf{V}(T_0)$ that belongs to the solved-for parameters of the optimization program.) However, we find that the time variation of V is fully negligible, at least over the time period investigated (one century).

D) Value of V. As was shown above, the equation of motion of a test particle in the gravitational field of a uniformly moving spherical object in SET, Eq. (64), coincides for $V = 0$ with Eq. (74). Now some ephemerides are based on Eq. (74) plus the Newtonian attractions due to the planets, as with Eqs. (78)–(79) of SET (and possibly including also oblateness corrections for the Sun's gravitational field): e.g. VSOP82 [23, 24], VSOP2000 [25], VSOP2013 [26]. (Also, the Warsaw ephemeris WAW [27] is based on the same approximation though it uses the Painlevé SSS solution of GR [28] instead of the spatially-isotropic SSS solution of GR.) As mentioned at Point A: in our equation of motion of a planet that includes the Newtonian attractions of the other planets, Eq. (78), more precisely in $\mathbf{h}$ and h_V that enter this equation and that are given by the expressions (65) and (66), the velocity $\mathbf{V}$ is replaced by $\mathbf{u}_N$, the velocity of the Sun — because the Sun does play the role of the supposedly unique massive body in Eq. (64). Since the VSOP2000 ephemeris is based on Eq. (74) plus the Newtonian attractions of the planets, and since Eq. (64) coincides with (74) for $\mathbf{V} = \mathbf{0}$, it follows that the equations of motion for VSOP2000 and for the present calculation are equivalent when $\mathbf{u}_N = \mathbf{0}$. We ran a parameter optimization for the Sun and the eight major planets, aimed at best fit the data of the DE403 ephemeris of the JPL

[29] over one century. Among the solved-for parameters is the velocity vector $\mathbf{V}$. Now, the difference between VSOP2000 and DE403 is very small [25]. [9] Therefore, one expects that the optimization program tries to make $\mathbf{u}_N$ as close as possible to $\mathbf{0}$, because then the equations of motion of SET are equivalent to those used in VSOP2000, that give nearly the same result as DE403. This means that $\mathbf{V}$, now defined as the velocity of the barycenter, should be found by the optimization program to be as close as possible to the velocity of the barycenter with respect to the Sun. The latter velocity varies in time but is approximately 40 km/h. And indeed, we find the modulus V (or equivalently V_0) to be of the order of 40 km/h. We believe that the foregoing discussion shows that this result is fully expected.

E) Comments. Nevertheless, the latter finding does not imply that SET has an accurate celestial mechanics only if V, the absolute velocity of the barycenter of the solar system, is nearly equal to zero — the latter being of course difficult to accept given what we know about galactic motion. It does not imply that mainly for two reasons:

i) As we know, the ephemerides are not direct observations but are a fitting of the true observations, which are diverse in nature [31], by the equations of motion based on the standard PN approximation of GR (essentially Eq. (74) plus the the Newtonian attractions of the planets — see Point D above). Moreover even these observations themselves, or at least many among them, are analysed and corrected precisely by using ephemerides, or more generally by using GR. For instance, the reduction of ranging data does use ephemerides [30]. This means that the observational data are in fact influenced by the theory (or more precisely by the approximate equations by which it is replaced in practice) that is used to "reduce" them — in the present case GR, or more precisely its standard PN approximation.

ii) The equations for extended bodies got for v1 had strong structure effects (including an effect of the self-rotation) and it is likely that something similar will apply to v2. Thus the correct equations of motion of the planets

[9] In the accurate ephemerides such as DE403 and followers, VSOP2000 and followers, WAW, EPM [30], etc.: the numerical precision is greater than in the present calculation; more bodies are taken into account: Pluto and many asteroids; etc. Hence the present calculation cannot aim at a similar accuracy. E.g. the longitude differences with DE403 are here at the 10 mas level over one century, except for Mercury (0.2 arcsec).

are quite different from those for test particles orbiting the Sun. Due to this difference, the correct 1PN equations of motion for extended bodies do not coincide with Eq. (74) [plus the Newtonian attractions due to the planets, as in Eqs. (78)–(79)] for $V = 0$. As a matter of fact, it had been found higher velocities (of the order of a few km/s) while fitting the corresponding equations of v1 to the DE403 ephemeris [10].

7 Conclusion

By studying in detail the equations of motion of a test particle in the investigated theory (SETv2), we have been able to put them in a tractable form. This allowed us to implement a first version of celestial-mechanical equations of motion for that theory in a software for ephemeris calculation with parameter optimization. Those simplified equations of motion coincide with equations used in the celestial mechanics of GR, when the absolute velocity of the Sun is zero. Therefore, they can lead to an equivalent celestial mechanics — but, with an unrealistically small velocity V for the barycenter of the solar system. To be able to really check what the theory says about V, one will need to develop a more realistic PN approximation, taking into account the self fields. Above all, one will need to make comparison with "direct" observations instead of ephemerides, and preferably with the observations being "reduced" (corrected) by using the investigated theory instead of GR. Especially the latter will be a hard specialized work.

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