Classical-quantum Correspondence and Wave Packet Solutions of the Dirac Equation in a Curved Spacetime

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Context of this work

- Long-standing problems with quantum gravity may mean: we should try to better understand (gravity, the quantum, and)

  the transition between classical and quantum, especially in a curved spacetime

- Quantum effects in the classical gravitational field are observed on spin $\frac{1}{2}$ particles $\Rightarrow$ Dirac eqn. in a curved ST

Foregoing work

- Analysis of classical quantum-correspondence: results from
  - An exact mathematical correspondence (Whitham): wave linear operator $\longleftrightarrow$ dispersion polynomial
  - de Broglie-Schrödinger idea: a classical Hamiltonian describes the skeleton of a wave pattern

  (M.A.: *il Nuovo Cimento B* **114**, 71–86, 1999)

- Led to deriving Dirac eqn from classical Hamiltonian of a relativistic test particle in an electromagnetic field or in a curved ST

- In a curved ST, this derivation led to 2 alternative Dirac eqs, in which the Dirac wave function is a complex four-vector

  (M.A.: *Found. Phys. Lett.* **19**, 225–247, 2006;
  Found. Phys. **38**, 1020–1045, 2008)

Foregoing work (continued)

- The quantum mechanics in a Minkowski spacetime in Cartesian coordinates is the same whether
  - the wave function is transformed as a spinor and the Dirac matrices are left invariant (standard transformation for this case)
  - or the wave function is a four-vector, with the set of Dirac matrices being a $(2\ 1)$ tensor ("TRD", tensor representation of Dirac fields)

  (M.A. & F. Reifler: *Brazil. J. Phys.* **38**, 248–258, 2008)

- In a general spacetime, the standard eqn & the two alternative eqs based on TRD behave similarly: e.g. same hermiticity condition of the Hamiltonian, similar non-uniqueness problems of the Hamiltonian theory
  (M.A. & F. Reifler: *Brazil. J. Phys.* **40**, 242–255, 2010;
  M.A. & F. R.: *Ann. der Phys.*, to appear in 2011)

Outline of this work

- Extension of the former derivation of the Dirac eqn from the classical Hamiltonian of a relativistic test particle: with an electromagnetic field and in a curved ST

- Conversely, from Dirac eqn to the classical motion through geometrical optics approximation:
  - The general Dirac Lagrangian in a curved spacetime
  - Local similarity (or gauge) transformations
  - Reduction of the Dirac eqn to a canonical form
  - Geometrical optics approximation into the Dirac canonical Lagrangian
  - Classical trajectories
  - de Broglie relations

Dispersion equation of a wave equation

Consider a linear (wave) equation (e.g., of 2nd order):

$$\mathrm{P}\psi \equiv \ a_0(X)\psi + a_1^{\mu}(X)\partial_\mu\psi + a_2^{\mu\nu}(X)\partial_\mu\partial_\nu\psi = 0, \quad (1)$$

where $X \leftrightarrow (ct, \mathbf{x}) =$ position in (configuration-)space-time.

Look for "locally plane-wave" solutions: $\psi(X) = A\,\exp[i\theta(X)]$,
with, at X_0, $\partial_\nu K_\mu(X_0) = 0$, where $K_\mu \equiv \partial_\mu\theta$.
$\mathbf{K} \leftrightarrow (K_\mu) \leftrightarrow (-\omega/c, \mathbf{k}) =$ wave covector.

Leads to the *dispersion equation:*

$$\Pi_X(\mathbf{K}) \equiv a_0(X) + i\,a_1^{\mu}(X)K_\mu + i^2 a_2^{\mu\nu}(X)K_\mu K_\nu = 0. \quad (2)$$

Substituting $K_\mu \hookrightarrow \partial_\mu/i$ determines the linear operator P *uniquely* from the polynomial function $(X, \mathbf{K}) \mapsto \Pi_X(\mathbf{K})$.

The classical-quantum correspondence

The *dispersion relation(s)*: $\omega = W(\mathbf{k}; X)$, fix the wave mode. Obtained by solving $\Pi_X(\mathbf{K}) = 0$ for $\omega \equiv -cK_0$. Witham: propagation of $\mathbf{k}$ obeys a *Hamiltonian system:*

$$\frac{\mathrm{d}K_j}{\mathrm{d}t} = -\frac{\partial W}{\partial x^j}, \qquad \frac{\mathrm{d}x^j}{\mathrm{d}t} = \frac{\partial W}{\partial K_j} \qquad (j = 1, ..., N). \tag{3}$$

Wave mechanics: a classical Hamiltonian H describes the skeleton of a wave pattern. Then, the wave eqn should give a dispersion W with the same Hamiltonian trajectories as H. Simplest way to get that: <u>assume</u> that H and W are proportional, $H = \hbar W$... *Leads first to* $E = \hbar\omega$, $\mathbf{p} = \hbar\mathbf{k}$, *or*

$$P_\mu = \hbar K_\mu \qquad (\mu = 0, ..., N) \qquad (= \text{de Broglie relations}). \tag{4}$$

Then, substituting $K_\mu \hookrightarrow \partial_\mu / i$, *it leads to the correspondence between a classical Hamiltonian and a wave operator.*

The classical-quantum correspondence needs using preferred classes of coordinate systems

The dispersion polynomial $\Pi_X(\mathbf{K})$ and the condition $\partial_\nu K_\mu(X)=0$ stay invariant only inside any class of "infinitesimally-linear" coordinate systems, connected by changes satisfying, at the point $X((x_0^\mu))=X((x_0'^\rho))$ considered,

$$\frac{\partial^2 x'^\rho}{\partial x^\mu \partial x^\nu}=0, \qquad \mu,\nu,\rho\in\{0,...,N\}. \tag{5}$$

One class: *locally-geodesic coordinate systems* at X for $\boldsymbol{g}$, *i.e.*,

$$g_{\mu\nu,\rho}(X)=0, \qquad \mu,\nu,\rho\in\{0,...,N\}. \tag{6}$$

Specifying a class $\Longleftrightarrow$ Choosing a *torsionless connection* D on the tangent bundle, and substituting $\partial_\mu \hookrightarrow D_\mu$.

A variant derivation of the Dirac equation

The motion a relativistic particle in a curved space-time derives from an "extended Lagrangian" in the sense of Johns (2005):

$$\mathcal{L}\left(x^{\mu}, u^{\nu}\right) = -mc\sqrt{g_{\mu\nu}u^{\mu}u^{\nu}} - (e/c)V_{\mu}u^{\mu}, \quad u^{\nu} \equiv dx^{\nu}/ds \tag{7}$$

The canonical momenta derived from this Lagrangian are

$$P_{\mu} \equiv \partial\mathcal{L}/\partial u^{\mu} = -mcu_{\mu} - (e/c)V_{\mu}. \tag{8}$$

They obey the following energy equation ($g^{\mu\nu}u_{\mu}u_{\nu} = 1$)

$$g^{\mu\nu}\left(P_{\mu} + \frac{e}{c}V_{\mu}\right)\left(P_{\nu} + \frac{e}{c}V_{\nu}\right) - m^2c^2 = 0, \tag{9}$$

Dispersion equation associated with this by wave mechanics:

$$g^{\mu\nu}\left(\hbar K_{\mu} + \frac{e}{c}V_{\mu}\right)\left(\hbar K_{\nu} + \frac{e}{c}V_{\nu}\right) - m^2c^2 = 0. \tag{10}$$

A variant derivation of the Dirac equation (continued)

Applying directly the correspondence $K_\mu \hookrightarrow D_\mu/i$ to the dispersion equation (10), leads to the Klein-Gordon eqn. Instead, one may try a *factorization:*

$$\Pi_X(\mathbf{K}) \equiv \left[g^{\mu\nu}(K_\mu + eV_\mu)(K_\nu + eV_\nu) - m^2\right]\mathbf{1}$$
$$=? \quad (\alpha + i\gamma^\mu K_\mu)(\beta + i\zeta^\nu K_\nu). \qquad (\hbar = 1 = c) \quad (11)$$

Identifying coeffs. (with noncommutative algebra), *and substituting* $K_\mu \hookrightarrow D_\mu/i$*, leads to the Dirac equation:*

$$(i\gamma^\mu(D_\mu + ieV_\mu) - m)\psi = 0, \qquad \text{with } \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1}. \quad (12)$$

General Dirac Lagrangian in a curved spacetime

The following Lagrangian (density) generalizes the "Dirac Lagrangian" valid for the standard Dirac eqn in a curved ST:

$$l = \sqrt{-g}\,\frac{i}{2}\left[\overline{\Psi}\gamma^\mu(D_\mu\Psi) - \left(\overline{D_\mu\Psi}\right)\gamma^\mu\Psi + 2im\overline{\Psi}\Psi\right], \qquad (13)$$

where $X \mapsto A(X)$ is the field of the *hermitizing matrix*:
$A^\dagger = A,\ (A\gamma^\mu)^\dagger = A\gamma^\mu$; and $\overline{\Psi} \equiv \Psi^\dagger A$ = adjoint of $\Psi \equiv (\Psi^a)$.

Euler-Lagrange equations $\rightarrow$ generalized Dirac equation:

$$\gamma^\mu D_\mu\Psi = -im\Psi - \frac{1}{2}A^{-1}(D_\mu(A\gamma^\mu))\Psi. \qquad (14)$$

Coincides with usual form iff $D_\mu(A\gamma^\mu) = 0$. Always the case for the standard, "Dirac-Fock-Weyl" (DFW) eqn.

Local similarity (or gauge) transformations

Given coeff. fields (γ^μ, A) for the Dirac equation, and given any *local similarity transformation* $S: X \mapsto S(X) \in \mathrm{GL}(4,\mathrm{C})$, other admissible coeff. fields are

$$\widetilde{\gamma}^\mu = S^{-1}\gamma^\mu S \quad (\mu = 0, ..., 3), \qquad \widetilde{A} \equiv S^\dagger A S. \tag{15}$$

The Hilbert space scalar product $(\Psi \mid \Phi) \equiv \int \Psi^\dagger A \gamma^0 \Phi \sqrt{-g}\, d^3\mathbf{x}$ transforms isometrically under the gauge transformation (15), if one transforms the wave function according to $\widetilde{\Psi} \equiv S^{-1}\Psi$.

The Dirac equation (14) is covariant under the similarity (15), if the connection matrices change thus:

$$\widetilde{\Gamma}_\mu = S^{-1}\Gamma_\mu S + S^{-1}(\partial_\mu S). \tag{16}$$

Reduction of the Dirac eqn to canonical form

If $D_\mu(A\gamma^\mu) = 0$ *and* the Γ_μ's are zero, the Dirac eqn (14) writes

$$\gamma^\mu \partial_\mu \Psi = -im\Psi. \tag{17}$$

Theorem 1. *Around any event X, the Dirac eqn (14) can be put into the canonical form (17) by a local similarity transformation.*

Outline of the proof: i) A similarity T brings the Dirac eqn to "normal" form ($D_\mu(A\gamma^\mu) = 0$), iff

$$A\gamma^\mu D_\mu T = -(1/2)[D_\mu(A\gamma^\mu)]T. \tag{18}$$

ii) A similarity S brings a normal Dirac eqn to canonical form, iff

$$A\gamma^\mu \partial_\mu S = -A\gamma^\mu \Gamma_\mu S. \tag{19}$$

Both (18) and (19) are symmetric hyperbolic systems. □

Geometrical optics approx. into Dirac Lagrangian

Lagrangian for the canonical Dirac equation in an e.m. field:

$$l = \sqrt{-g}\,\frac{i\hbar c}{2}\left[\Psi^\dagger A\gamma^\mu(\partial_\mu\Psi) - (\partial_\mu\Psi)^\dagger A\gamma^\mu\Psi + \frac{2imc}{\hbar}\Psi^\dagger A\Psi\right] - -\sqrt{-g}\,(e/c)J^\mu V_\mu \quad (20)$$

with $\nabla_\mu(A\gamma^\mu) = 0$. Substitute $\Psi = \chi e^{i\theta}$ with $\partial_\mu\chi \ll (\partial_\mu\theta)\chi$:

$$l' = c\sqrt{-g}\left[\left(-\hbar\partial_\mu\theta - \frac{e}{c}V_\mu\right)\chi^\dagger A\gamma^\mu\chi - mc\chi^\dagger A\chi\right] \quad (21)$$

Euler-Lagrange eqs:

$$\left(-\hbar\partial_\mu\theta - \frac{e}{c}V_\mu\right)A\gamma^\mu\chi = mcA\chi \quad (22)$$

$$\partial_\mu\left(c\sqrt{-g}\,\chi^\dagger A\gamma^\mu\chi\right) = 0 \quad (23)$$

Classical trajectories

Theorem 2. *From* $\Psi = \chi e^{i\theta}$*, define a four-vector field* u^μ *and a scalar field* J *thus:*

$$u_\mu \equiv -\frac{\hbar}{mc}\partial_\mu \theta - \frac{e}{mc^2}V_\mu, \tag{24}$$

$$u^\mu \equiv g^{\mu\nu}\, u_\nu, \tag{25}$$

$$J \equiv c\, \chi^\dagger A \chi. \tag{26}$$

Then the Euler-Lagrange eqs (22) imply

$$\nabla_\mu (J u^\mu) = 0, \tag{27}$$

$$g^{\mu\nu}\, u_\mu u_\nu = 1, \tag{28}$$

$$\nabla_\mu u_\nu - \nabla_\nu u_\mu = -(e/mc^2)\, F_{\mu\nu}. \tag{29}$$

The two last eqs imply the classical equation of motion for a test particle in an electromagnetic field in a curved spacetime.

De Broglie relations

Canonical momenta of a classical particle, Eq. (8):

$$P_\mu \equiv \partial \mathcal{L}/\partial u^\mu = -mcu_\mu - (e/c)V_\mu. \tag{30}$$

Definition (24) of a 4-velocity field u_μ from the phase θ of the wave function of a Dirac quantum particle:

$$u_\mu \equiv -\frac{\hbar}{mc}\partial_\mu\theta - \frac{e}{mc^2}V_\mu, \tag{31}$$

or (remembering the definition $K_\mu \equiv \partial_\mu\theta$):

$$-mcu_\mu - (e/c)V_\mu \equiv \hbar K_\mu. \tag{32}$$

Thus, we get the de Broglie relations:

$$\underline{P_\mu = \hbar K_\mu}. \tag{33}$$

Conclusion

- The Dirac eqn *in a curved spacetime with electromagnetic field* may be "derived" from the classical Hamiltonian H of a relativistic test particle. One has to postulate $H = \hbar W$ where W is the dispersion relation of the sought-for wave eqn, and to factorize the obtained dispersion polynomial.

- Conversely, to describe "wave packet" motion: implement the geometrical optics approximation into a canonical form of the Dirac Lagrangian. From the eqs obtained thus for the amplitude and phase of the wave function, one defines a 4-velocity u^μ. This obeys exactly the classical eqs of motion.

  The de Broglie relations $P_\mu = \hbar K_\mu$ are then derived exact eqs.