Dirac equation in a curved spacetime: the standard equation and the alternatives

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Outline

- Transformation of Dirac eqn in Minkowski spacetime
- General spacetime: a common geometrical framework
- Three classes of Dirac equations in a general spacetime
  - The standard ("Dirac-Fock-Weyl") equation (in some detail)
  - Two interesting alternative equations
- Non-uniqueness of the Hamiltonian & energy operators
- Conclusion

Dirac equation in the flat Minkowski spacetime M

Let $\chi : \mathrm{M} \to \mathbb{R}^4, \quad X \mapsto \mathbf{X} \equiv (x^\mu)$ be a global Cartesian coordinate system on M. In such a system, Dirac's original eqn writes:

$$\underline{\gamma^\mu \, \partial_\mu \Psi = -im\Psi}. \tag{1}$$

$\gamma^\mu \ (\mu = 0, ..., 3)$: Dirac matrices: any quadruplet of invertible 4×4 complex matrices verifying the anticommutation relation

$$[\gamma^\mu, \gamma^\nu]_+ \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \, \mathbf{1}_4, \quad (\eta^{\mu\nu}) \equiv \mathrm{diag}(1, -1, -1, -1). \tag{2}$$

$\Psi : \mathbb{R}^4 \to \mathbb{C}^4, \ \mathbf{X} \equiv (x^\mu) \mapsto \Psi(\mathbf{X})$: expression of the Dirac wave function in the Cartesian chart considered.

Transformation of the Dirac equation in the flat Minkowski spacetime M

Let $\chi : \mathsf{M} \to \mathbb{R}^4$ be a global chart. Let $\Psi : \mathbb{R}^4 \to \mathbb{C}^4$, $\quad \mathbf{X} \mapsto \Psi(\mathbf{X})$ be the expression of the Dirac wave function ψ in the chart χ.

One asks that after certain linear chart changes: $\chi \hookrightarrow \chi' = L.\chi$ with $L \in \mathrm{G}$, where G is a subgroup of $\mathsf{GL}(4, \mathbb{R})$, Ψ become Ψ' such that

$$\Psi'(\mathbf{X}') = S.\Psi(\mathbf{X}), \qquad S = \mathsf{S}(L) \in \mathsf{GL}(4, \mathbb{C}), \tag{3}$$

for some function S of the 4×4 matrix $L \in \mathrm{G}$. (More detail here.)

This occurs if and only if S is a *representation* $\mathrm{G} \to \mathsf{GL}(4, \mathbb{C})$.
(MA, Found. Phys. Lett. **19**, 225, 2006)

Transforming the Dirac eqn in the M ST (continued)

Let's start from a Cartesian chart, thus we may write the flat Dirac eqn (1). After a chart change $L \in \mathrm{G}$, (1) becomes

$$\gamma'^{\nu}\, \partial'_{\nu} \Psi' = -im\Psi', \quad \gamma'^{\nu} \equiv L^{\nu}{}_{\mu}\, S\gamma^{\mu} S^{-1}, \quad S \equiv \mathsf{S}(L). \tag{4}$$

Usual statement: "*Relativity* $\Rightarrow \gamma'^{\nu} = \gamma^{\nu}$." (Then one gets for S the "spinor representation".) <u>NO.</u> Archetyp of a relativist eqn: eqn of motion of a particle with 4-velocity U^{μ} in the electromagnetic field F^{μ}_{ν}:

$$m\frac{\mathrm{d}U^{\mu}}{\mathrm{d}s} = qF^{\mu}_{\nu}U^{\nu}, \qquad \text{or} \quad m\frac{\mathrm{d}U}{\mathrm{d}s} = qFU. \tag{5}$$

The matrix $F \equiv (F^{\mu}{}_{\nu})$ is *not* invariant: $F' = LFL^{-1} \neq F$.

Dirac wave function as a 4-scalar or a 4-vector

Two simpler possibilities than the spinor representation for S:

- $\mathsf{S}(L) = L: \quad \Psi'(\mathbf{X}') = L.\Psi(\mathbf{X}), \quad \gamma'^{\mu} \equiv L^{\mu}{}_{\nu} L \gamma^{\nu} L^{-1}.$

  I.e., <u>the Dirac wave function is a 4-vector</u> and <u>the components $(\gamma^{\mu})^{\rho}{}_{\nu}$ make a $\binom{2}{1}$ tensor.</u>

  (MA, Found. Phys. Lett. 2006; Found. Phys. **38**, 1020, 2008)

- $\mathsf{S}(L) = \mathbf{1}_4: \quad \Psi'(\mathbf{X}') = \Psi(\mathbf{X}), \quad \gamma'^{\mu} \equiv L^{\mu}{}_{\nu} \gamma^{\nu}:$

  Dirac wave function is a 4-scalar and the set of the Dirac matrices transforms as a 4-vector. <u>This is the transformation law for the standard Dirac eqn in a curved S-T, Dirac-Fock -Weyl.</u> (MA & F. Reifler, Braz. J. Phys. **38**, 248, 2008.)

Transforming the Dirac eqn in the M ST (continued)

$\diamondsuit$ For any pair (G, S) with G a subgroup of $\mathsf{GL}(4, \mathbb{R})$ and S a representation of G into $\mathsf{GL}(4, \mathbb{C})$:

- The Dirac eqn (1) is covariant under all changes $\chi \hookrightarrow \chi' = L.\chi$ with $L \in \mathrm{G}$: see Eq. (4).

- The anticommutation relation (2) is covariant, too:

$$[\gamma'^{\mu}, \gamma'^{\nu}]_{+} = 2g'^{\mu\nu}\, \mathbf{1}_4 \tag{6}$$

  (with $g'^{\mu\nu} = g^{\mu\nu} = \eta^{\mu\nu}$ if $L \in \mathsf{O}(1, 3)$).

- The quantum mechanics associated with the "flat" Dirac eqn is the same: the *equation*, hence its *solutions*, are the same. (The choice of the quadruplet (γ^{μ}) has no effect on QM: MA & F. Reifler, Braz. J. Phys. 2008.)

Transforming the Dirac eqn in the $\mathbb{M}$ ST (end)

$\diamondsuit$ The spinor representation is restricted to $L \in \mathsf{SO}(1,3)$. Hence it cannot be used for a general chart in a general spacetime, that leads to a general matrix $L \equiv \left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right) \in \mathsf{GL}(4,\mathbb{R})$.

$\diamondsuit$ In contrast, the representations

- $\mathsf{S}(L) = \mathbf{1}_4$ (ψ 4-scalar)
- $\mathsf{S}(L) = L$ (ψ 4-vector)

hold valid for any $L \in \mathsf{GL}(4,\mathbb{R})$.

$\Rightarrow$ These two representations extend to a general spacetime.

General spacetime: a common, simple, geometrical framework

- Thus, in a general spacetime, the wave function ψ can be only a 4-scalar or a 4-vector. Depending on either choice, ψ is a section of a complex vector bundle with base V (the spacetime manifold), having rank 4, denoted E. Thus $\psi \in \Gamma(\mathsf{E}) : \ \mathrm{V} \ni X \mapsto \psi(X) \in \mathsf{E}_X$, with:
  - $\mathsf{E} = \text{trivial bundle } \mathrm{V} \times \mathbb{C}^4 \qquad\qquad \text{for } \ \psi \ \ 4-\text{scalar}$

    ("Quadruplet Representation of the Dirac field", **QRD**)
  - $\mathsf{E} = \text{complexified tangent bundle } \ \mathrm{T}_{\mathbb{C}}\mathrm{V} \qquad \text{for } \ \psi \ \ 4-\text{vector}$

    ("Tensor Representation of the Dirac field", **TRD**).

Geometrical framework (continued)

The "intrinsic field of Dirac matrices" γ lives in the tensor product $\mathrm{TV} \otimes \mathsf{E} \otimes \mathsf{E}^\circ$, where E° is the dual vector bundle of E.

The Dirac matrices γ^μ themselves are local and are made with the components of γ:

$$\gamma_{|\mathrm{U}} = \gamma_b^{\mu a} \partial_\mu \otimes e_a \otimes \theta^b \Rightarrow (\gamma^\mu)^a{}_b \equiv \gamma_b^{\mu a} \quad (a, b = 0, ..., 3). \qquad (7)$$

♢ They depend on the local coordinate basis (∂_μ) on an open subset U of the spacetime V, and on the local frame field (e_a) on E above U. ((θ^b) is the dual frame field on E° above U.)

♢ They have to verify ($\forall$ chart (χ, U) & $\forall$ frame field (e_a) on E above U) the anticommutation relation $[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}\, \mathbf{1}_4$.

Geometrical framework (continued)

- For QRD ($\mathsf{E} = \mathrm{V} \times \mathbb{C}^4$), the canonical basis of $\mathbb{C}^4$ is a preferred frame field on E, whence the scalar (=invariant) character of the wave function ψ.

- For TRD ($\mathsf{E} = \mathrm{T}_{\mathbb{C}}\mathrm{V}$), the frame field on E can be taken to be the coordinate basis (∂_μ). Then on changing the coordinate chart, ψ behaves as an usual four-vector, and γ as an usual $(2\ \ 1)$ tensor.

- Relations between QRD and TRD, and between Dirac eqs got with different connections, have been studied.

(MA–F. Reifler, Int. J. Geom. Meth. Mod. Phys. 1250026, 2012; summary: MA–FR, J. Phys. Conf. Ser. **306**, 012061, 2011.)

Geometrical framework (end)

- Usually, mathematicians start from a “spinor structure” on V (see e.g. Isham, Proc. Roy. Soc. A **364**, 591, 1978).

  If the spacetime V (is 4-dimensional, non-compact, and) admits a spinor structure, then there exists a global orthonormal tetrad field (u_α) on TV (Geroch, J. Math. Phys. **9**, 1739, 1968).

  In that case, each of the two very simple vector bundles E defined above is a “spinor bundle”, i.e., there exists a global “intrinsic field of Dirac matrices” $\gamma \in \Gamma(\mathrm{TV} \otimes \mathsf{E} \otimes \mathsf{E}^\circ)$ verifying (a coordinate-free form of) the anticommutation relation (MA–FR, IJGMMP 2012).

The Dirac equation and the choices of it

Choose

- the representation: E $= \mathrm{V} \times \mathbb{C}^4$ (QRD) or E $= \mathrm{T}_{\mathbb{C}}\mathrm{V}$ (TRD);
- any "intrinsic field of Dirac matrices", γ, i.e., any section of $\mathrm{TV} \otimes$ E $\otimes$ E° verifying the anticommutation relation;
- any connection on E, $D : \psi \mapsto D\psi$. (In a frame field (e_a) on E, $\psi = \Psi^a e_a$, and D is given by connection matrices Γ_μ, such that $D_\mu \Psi \equiv (D_\mu \Psi^a) = (\partial_\mu + \Gamma_\mu)\Psi$, where $\Psi \equiv (\Psi^a)$).

Then, only one Dirac equation may be written (MA–FR, IJGMMP 2012):

$$\gamma : D\psi \left(= \gamma^{\mu a}_{b}\, D_\mu \Psi^b\, e_a\right) = -im\psi, \tag{8}$$

but it depends on each of the three choices...

Three classes of Dirac equations

- I. The standard, Dirac-Fock-Weyl eqn, is a QRD eqn got when one assumes (Brill–Wheeler, Rev. Mod. Phys. **29**, 465, 1957; Chapman–Leiter, Am. J. Phys. **44**, 858, 1976; MA–FR, IJGMMP 2012) that:
  - the field γ is deduced from some set $(\gamma^{\sharp\alpha})$ of "flat" Dirac matrices and from some global *tetrad field* (u_α) on TV, i.e. a field of direct orthonormal bases of TV :

$$\gamma = (\gamma^{\sharp\alpha})^a{}_b \, u_\alpha \otimes E_a \otimes \Theta^b \tag{9}$$

    ((E_a) = canonical basis of $\mathbb{C}^4$, (Θ^a) = dual base);
  - the connection D on $\mathrm{V} \times \mathbb{C}^4$ depends on γ in such a way that $D\gamma = 0$.

The standard equation, DFW (continued)

- Remark 1. The condition $D\gamma = 0$ (i.e. $D_\nu \gamma_b^{\mu a} = 0$) is *not* imposed by the conjunction of $g_{\mu\nu;\rho} = 0$ and of the anticommutation (2). However, it ensures easily that one has
$$D_\mu(A\gamma^\mu) \equiv (D_\mu(A_{ac}\,\gamma_b^{\mu c}))_{a,b=0,...,3} = 0, \qquad (10)$$
with A the "hermitizing matrix" (usually $A = \gamma^{\sharp 0}$ for DFW). The condition (10) is necessary and sufficient in order that the solutions of the curved-S-T Dirac eqn (8) all obey the current conservation (MA–F. Reifler, Braz. J. Phys. **40**, 242, 2010).

- Rmk 2. The condition $D\gamma = 0$ leads to the explicit expression of the matrices Γ_μ of the "spin connection" D on $\mathsf{E} = \mathrm{V} \times \mathbb{C}^4$, as function of the tetrad field (u_α) — up to the addition of a term $\lambda \mathbf{1}_4$ (Chapman–Leiter 1976).

The standard equation, DFW (continued)

- <u>Rmk 3.</u> Any two tetrad fields (u_α) and $(\widetilde{u}_\alpha)$ are related together by a "local Lorentz transformation": a (global!) smooth map

$$L = (L^\alpha{}_\beta) : \mathrm{V} \to \mathsf{SO}(1,3)\,; \quad \widetilde{u}_\beta = L^\alpha{}_\beta \, u_\alpha . \tag{11}$$

  At least locally (on $\mathrm{U} \subset \mathrm{V}$), L can be "lifted" to a smooth mapping $S : \mathrm{U} \to \mathsf{Spin}(1,3)$ such that $\Lambda \circ S = L$, with $\Lambda : \mathsf{Spin}(1,3) \to \mathsf{SO}(1,3)$ the two-to-one cover of $\mathsf{SO}(1,3)$. If V is simply connected this occurs globally (e.g. Isham 1978).

  In U we have $\underline{\widetilde{\gamma}^\mu = S^{-1}\gamma^\mu S}$, and (defining $\widetilde{\Psi} \equiv S^{-1}\Psi$) <u>the DFW eqs with the fields (γ^μ, Ψ) and $(\widetilde{\gamma}^\mu, \widetilde{\Psi})$ are equivalent.</u>

The standard equation, DFW (continued)

- Rmk 4. In the physics literature on the DFW eqn (e.g. in Brill–Wheeler 1957 or Chapman–Leiter 1976) this covariance of the DFW eqn under a change of the tetrad field is stated, not derived. In the maths literature on "spin geometry", the argument goes *grosso modo* like this (see Isham 1978) :

  The spin connection D on the spinor bundle E is got uniquely from the Levi-Civita connection, *via* i) its extension to the principal bundle O of orthonormal frames and ii) the spinor structure. D depends on the spinor structure (thus essentially on the tetrad field), but equivalent spinor structures give rise to "gauge-related" connections.

The standard equation, DFW (continued)

- The idea of "lifting" the Levi-Civita connection (expressed in terms of parallel transport) was in the initial works (Weyl, Proc. Nat. Ac. Sci. **15**, 323, 1929; Fock, J. Phys. Rad. **10**, 392, 1929). Cf. Scholz, physics/0409158.

- Rmk 5. That the DFW eqn is covariant under a change of the tetrad field: $(u_\alpha) \hookrightarrow (\widetilde{u}_\alpha)$, has ultimately to be checked by explicit computation.

  One has to verify that, S s.t. $\Lambda \circ S = L$ being defined as above: on the simultaneous changes $\widetilde{\gamma}^\mu = S^{-1}\gamma^\mu S$ and $\widetilde{\Psi} \equiv S^{-1}\Psi$, the connection matrices are "gauge-related" as follows:

$$\widetilde{\Gamma}_\mu = S^{-1}\Gamma_\mu S + S^{-1}(\partial_\mu S). \tag{12}$$

  This has indeed been checked (Fock, J. Phys. Rad. 1929).

The standard equation, DFW (end)

- Rmk 6. In the present framework, the transformation S: $\gamma^\mu \hookrightarrow \widetilde{\gamma}^\mu$, $\Psi \hookrightarrow \widetilde{\Psi}$, $\Gamma_\mu \hookrightarrow \widetilde{\Gamma}_\mu$ (with $\Lambda \circ S = L$), resulting from the change $(u_\alpha) \hookrightarrow (\widetilde{u}_\alpha)$, is seen as an *active* local similarity transformation in the sense of (MA–FR, IJGMMP 2012):

  The frame field on E $= \mathrm{V} \times \mathbb{C}^4$ is not changed by the transformation. It remains the canonical basis (E_a), as in Eq. (9) — which is valid also after the similarity, though with $\widetilde{u}_\alpha$ in the place of u_α, hence with a new γ field, say $\widetilde{\gamma}$. Also:

  — $\widetilde{\Psi}^a$ $(a = 0, ..., 3)$ are the components, in the frame field (E_a), of a new wave function $\widetilde{\psi} = \widetilde{\Psi}^a E_a \in \Gamma(\mathsf{E})$.

  — $\widetilde{\Gamma}_\mu$ $(\mu = 0, ..., 3)$ are the connection matrices, in the frame fields (∂_μ) (on TV) and (E_a) (on E), of a new connection $\widetilde{D}$ on E.

3 classes of Dirac equations (continued)

- II. The QRD–0 eqs assume that $D\,E_a = 0$, where (E_a) is the canonical basis of $\mathbb{C}^4$, seen as a constant frame field on $\mathrm{V} \times \mathbb{C}^4$.

  I.e., the connection matrices are zero: $D_\mu \Psi^a = \partial_\mu \Psi^a$!
  (MA–FR, IJGMMP 2012)

- III. The TRD–1 eqs assume the Levi-Civita connection, extended trivially from TV to $\mathsf{E} = \mathrm{T}_{\mathbb{C}}\mathrm{V}$.
  In contrast with DFW, this version is fully compatible with the equivalence principle. (MA, Found. Phys. 2008)

For each of those 2: the connection D is fixed. The γ field can still (but need not) be defined from a tetrad field. Generally, two fields $\gamma \neq \gamma'$ give inequivalent Dirac eqs. Hence: *classes* of eqs.

The general Dirac Hamiltonian

- Rewriting the covariant Dirac eqn (8) in Schrödinger's form: $i\frac{\partial \Psi}{\partial t} = \mathrm{H}\Psi \quad (t \equiv x^0)$, gives the general explicit expression of the Dirac Hamiltonian operator H.

  (Parker, Phys. Rev. D **22**, 1922, 1980; MA–F. Reifler, Ann. der Phys. **523**, 531, 2011)

- The Hamiltonian depends naturally on the coordinate system, or more exactly on the *reference frame* — an equivalence class of charts defined on a given open set $\mathrm{U} \subset \mathrm{V}$ and exchanging by

$$x'^0 = x^0, \quad x'^j = f^j((x^k)) \qquad (j, k = 1, 2, 3). \tag{13}$$

  (MA–F. Reifler, Braz. J. Phys. 2010; Int.J.Geom.Meth.Mod.Phys. **8**, 155, 2011. Thus a chart χ defines a reference frame: the equivalence class of χ.)

The Dirac Hamiltonian is Non-Unique

(MA–FR, Ann. der Phys. 2011)

- It has been found that the Dirac Hamiltonian operator H *in a given chart* is not unique: it depends on the admissible choice of the field of Dirac matrices. For DFW, it means: on the tetrad field. (See also MA, Int. J. Theor. Phys. **52**, 4032, 2013.) *Idem* for the energy operator E (the Hermitian part of H for the relevant scalar product).

- The *spectrum* of E is itself non-unique. All of this applies already in an inertial frame in a flat spacetime, and also if there is an external electromagnetic field. So, in contrast with Dirac's original eqn, the covariant Dirac eqn (8) can't predict the energy levels of the hydrogen atom!! (MA, Int. J. Theor. Phys. **53**, 2014, to appear.)

- This is true for all versions of the covariant Dirac eqn (8).

Summary

- Spinor transformation <u>not</u> needed to ensure the Lorentz covariance of the Dirac eqn of special relativity. Wave function can also be defined as 4-scalar or as 4-vector.

- In a curved spacetime, the Dirac wave function can <u>only</u> be defined as a 4-scalar, <u>or</u> as a 4-vector.

- Standard version of (generally-)covariant Dirac eqn: Dirac-Fock-Weyl eqn. Its big advantage: it is unique (in a topologically-simple spacetime).

- But, there is a severe non-uniqueness problem of the Hamiltonian and energy operators.

Concluding remarks

- To solve the non-uniqueness problem of the Hamiltonian and energy operators, one needs to restrict the gauge freedom — for DFW, this means to restrict the choice of the tetrad field.

- When the necessary restriction is implemented, the alternative eqn "QRD–0" becomes unique, as well as is DFW (MA, Int. J. Geom. Meth. Mod. Phys. **10**, 1350027, 2013).

Slide 4, more precise

Let $\chi_0 : \mathsf{M} \rightarrow \mathbb{R}^4$ be a global chart of M; G a subgroup of $\mathsf{GL}(4, \mathbb{R})$; $\bar{\mathrm{G}}$ the set of the charts $\chi = L.\chi_0$ for some $L \in \mathrm{G}$: $\chi : X \mapsto \mathbf{X} = L.\mathbf{X}_0 \in \mathbb{R}^4$, where $\mathbf{X}_0 \equiv \chi_0(X) \in \mathbb{R}^4$.

$\forall$ chart $\chi \in \bar{\mathrm{G}}$, let $\mathbf{X} \mapsto \Psi_\chi(\mathbf{X}) \in \mathbb{C}^4$ be the expression of the Dirac wave function ψ in the chart χ and in some fixed frame field (e_a) on the vector bundle E, of which ψ is a section.

One asks that, on a change $\chi \hookrightarrow L.\chi$, the transformation of $\Psi_\chi(\mathbf{X})$ be linear and depend on a function $\mathsf{S} : \mathrm{G} \rightarrow \mathsf{GL}(4, \mathbb{C})$ in such a way that

$$\forall \chi \in \bar{\mathrm{G}}, \ \ \forall L \in \mathrm{G}, \ \ \forall \mathbf{X} \in \mathbb{R}^4, \ \ \Psi_{L.\chi}(L.\mathbf{X}) = \mathsf{S}(L).\Psi_\chi(\mathbf{X}). \quad (14)$$

This happens if and only if S is a *representation* $\mathrm{G} \rightarrow \mathsf{GL}(4, \mathbb{C})$.