Representations of the Dirac wave function in a curved spacetime

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Context of this work

- Quantum effects in the classical gravitational field *are observed,* e.g. on neutrons: spin $\frac{1}{2}$ particles.
  $\Rightarrow$ Motivates work on the curved spacetime Dirac eqn
- Minkowski spacetime: under a Lorentz transformation, the Dirac wave function ψ transforms under the spin group, while the Dirac matrices γ^μ are left invariant
- This is not an option in a curved spacetime or in general coordinates in a flat ST: the spinor representation does not extend to the linear group
- Standard "Dirac eqn in a curved ST": Dirac-Fock-Weyl eqn. In it, $\psi \equiv (\psi^a)$ transforms as a quadruplet of complex scalars and the set of the γ^μ 's transforms as a four-vector

Foregoing work

- Tensor representation of Dirac field (**TRD**):
  - Wave function ψ is a complex four-vector
  - Set of components of Dirac matrices γ^μ builds a $(2\ \ 1)$ tensor

  (M.A.: *Found. Phys. Lett.* **19**, 225–247, 2006)

- In a flat ST in Cartesian coordinates, the three representations of ψ (spinor, scalar, vector) lead to the *same quantum mechanics*
  (M.A. & F. Reifler: *Braz. J. Phys.* **38**, 248–258, 2008)

- In a curved ST, two alternative Dirac eqs proposed, based on TRD
  (M.A.: *Found. Phys.* **38**, 1020–1045, 2008)

- The standard eqn & the two alternative eqs based on TRD behave similarly: e.g. same hermiticity condition of the Hamiltonian, similar non-uniqueness problems of the Hamiltonian theory in a curved ST
  (M.A. & F. Reifler: *Braz. J. Phys.* **40**, 242–255, 2010)

Outline of the present work

The similar behaviour we found for the Dirac-Fock-Weyl eqn (with ψ 4-scalar) and our alternative eqs based on TRD led us to study the *relations between the two representations in a curved ST* (ψ 4-scalar vs. ψ 4-vector). In the present study:

- The two representations were formulated in a common geometrical framework
- Equivalence theorems were proved between different representations & between different classes of eqns

A common geometrical framework

- Dirac-Fock-Weyl eqn belongs to the more general "quadruplet representation of the Dirac field" (**QRD**)

- For both QRD and the tensor representation (TRD), the wave function lives in some complex vector bundle with base V (the spacetime manifold), and with dimension 4, denoted E:

  - $\mathrm{E} = \text{trivial vector bundle } \mathrm{V} \times \mathsf{C}^4$ for QRD
  - $\mathrm{E} = \text{ complexified tangent bundle } \mathrm{T}_{\mathsf{C}}\mathrm{V}$ for TRD

- Other relevant objects (e.g. the field of Dirac matrices) also expressed using E.

Geometrical framework (continued)

The "intrinsic field of Dirac matrices" γ lives in the tensor product $\mathrm{TV} \otimes \mathrm{E} \otimes \mathrm{E}^\circ$, where E° is the dual vector bundle of E.

The Dirac matrices γ^μ themselves are made with the components of γ:

$$(\gamma^\mu)^a{}_b \equiv \gamma^{\mu a}_b. \qquad (1)$$

They depend on the local coordinate basis (∂_μ) on the spacetime V, on the local frame field (e_a) on E, and on the associated dual frame field (θ^b) on E°.

Geometrical framework (end)

- For QRD ($\mathrm{E} = \mathrm{V} \times \mathrm{C}^4$), the canonical basis of C^4 is a preferred frame field on E, whence the scalar (=invariant) character of the wave function ψ.

- For TRD ($\mathrm{E} = \mathrm{T}_\mathrm{C}\mathrm{V}$), the frame field on E can be taken to be the coordinate basis (∂_μ).

  Then on changing the coordinate chart, ψ behaves as an usual four-vector, and γ as an usual $(2\;\;1)$ tensor.

The Dirac equation and the choices of it

Choose

- the representation, i.e., $\mathrm{E} = \mathrm{V} \times \mathbb{C}^4$ or $\mathrm{E} = \mathrm{T}_{\mathbb{C}}\mathrm{V}$;
- any "intrinsic field of Dirac matrices", γ, i.e., any section of $\mathrm{TV} \otimes \mathrm{E} \otimes \mathrm{E}^{\circ}$ so that the associated Dirac matrices γ^μ (that depend on the chart and the frame field) satisfy the (covariant) anticommutation relation $[\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu}\mathbf{1}_4$;
- any connection $D : \psi \mapsto D\psi$ on E.

Then only one Dirac equation may be written:

$$\gamma : D\psi \left(= \gamma_b^{\mu a} (D\psi)_\mu^b \, e_a\right) = -im\psi, \qquad (2)$$

but it depends on each of the three choices...

Four classes of Dirac equations

- 1) The standard, Dirac-Fock-Weyl eqn, obtains when one assumes that:
  - the field γ is deduced from some real *tetrad field;*
  - the connection D on $\mathrm{V} \times \mathbb{C}^4$ depends on γ so that $D\gamma = 0$.

NB: Any two tetrad fields lead to two equivalent Dirac-Fock-Weyl eqs (except for non-trivial topologies).

4 classes of Dirac equations (continued)

- 2) The QRD–0 eqs assume that $D\,E_a = 0$, where (E_a) is the canonical basis of $\mathrm{V} \times \mathbb{C}^4$.

- 3) The TRD–0 eqs assume that $D\,e_a = 0$, where (e_a) is some global orthonormal frame field (tetrad field) on $\mathrm{T}_\mathbb{C}\mathrm{V}$.

- 4) The TRD–1 eqs assume the Levi-Civita connection, extended from TV to $\mathrm{T}_\mathbb{C}\mathrm{V}$.

For each of those three: the connection D is fixed, but the field γ is restricted only by the anticommutation relation. In general, two fields $\gamma \neq \gamma'$ give inequivalent Dirac eqs.

Equivalence theorems between classes

1) QRD–0 and TRD–0 are equivalent for a given γ^μ field. (easy)

2) Let γ be any "intrinsic field of Dirac matrices" and let D be any connection on E. Let D' be any (other) connection on E.

There is another "intrinsic field", $\tilde{\gamma}$, such that the Dirac eqn based on γ and D is equivalent to that based on $\tilde{\gamma}$ and D'.

In particular, any form of the QRD (TRD) eqn is equivalent to a QRD–0 (TRD–1) eqn.

3) 1 + 2 $\Rightarrow$ *The Dirac-Fock-Weyl eqn is equivalent to a TRD–1 eqn (thus* **with vector wave function***) in the same spacetime.*

Theorem 2: outline of the proof

For a given field γ, the difference between the Dirac operators $\mathcal{D}(\gamma, D)$ and $\mathcal{D}(\gamma, D')$ is found to depend just on the matrix

$$K \equiv \gamma^\mu K_\mu, \tag{3}$$

where the γ^μ 's are the Dirac matrices associated with γ in the local chart and frame field considered, and with

$$K_\mu \equiv \Gamma_\mu - \Gamma'_\mu, \tag{4}$$

Γ_μ and Γ'_μ being the connection matrices of D and D'.

Consider a new field $\tilde{\gamma}$. We know how to change D for a new connection $\tilde{D}$ so that $\mathcal{D}(\gamma, D)$ is equivalent to $\mathcal{D}(\tilde{\gamma}, \tilde{D})$. Set $\tilde{K}_\mu \equiv \tilde{\Gamma}_\mu - \Gamma'_\mu$ and $\tilde{K} \equiv \tilde{\gamma}^\mu \tilde{K}_\mu$. If $\tilde{K} = 0$, the Dirac operator $\mathcal{D}(\tilde{\gamma}, D')$ is equivalent to $\mathcal{D}(\tilde{\gamma}, \tilde{D})$, hence to $\mathcal{D}(\gamma, D)$.

Theorem 2: outline of the proof (end)

Let a local similarity transformation $\mathrm{V} \ni X \mapsto S(X) \in \mathsf{GL}(4, \mathsf{C})$ lead to a new field of Dirac matrices:

$$\tilde{\gamma}^{\mu}(X) \equiv S(X)^{-1}\gamma^{\mu}(X)S(X) \tag{5}$$

The condition for $\tilde{K} \equiv \tilde{\gamma}^{\mu}\tilde{K}_{\mu} = 0$ is then

$$\gamma^{\mu}D'_{\mu}S = -KS. \tag{6}$$

This is a system of sixteen first-order linear partial differential equations for the sixteen components of S, which can be rewritten as a symmetric hyperbolic system. Therefore, by known theorems, this can be solved. □