Dirac equation in a gravitational field: wave mechanics vs equivalence principle

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Quantum Mechanics (QM) in a gravitational field: ?

- Gravity is described by theories with curved space-time.
- Usual way to write QM wave eqs: covariantization (or *equivalence principle*): searched eq. in curved S-T should coincide with flat-S-T version in a local freely falling frame. But ambiguities may arise due to non-commuting covariant derivatives. E.g.: covariant Klein-Gordon eq**s**.
- Alternatively, one may try to *apply directly the classical-quantum correspondence.* But:
  - Is there a classical Hamiltonian?
  - Canonical quantization will depend on coordinates.
  - Even in given coordinates, ambiguities arise because metric contains position which doesn't commute with momentum.

Dispersion equation of a linear wave eqn

Whitham's theory of dispersive linear waves, a bit extended.

A linear (wave) eq.: $a_0(X)\psi + a_1^{\mu}(X)\partial_\mu\psi + a_2^{\mu\nu}(X)\partial_\mu\partial_\nu\psi = 0$, where $X = (t, \mathbf{x}) =$ position in (configuration-)space-time.

Look for "locally plane-wave" solutions: $\psi(X) = A\,\exp[i\theta(X)]$, with, at X_0, $\partial_\nu\partial_\mu\theta(X_0) = \partial_\nu K_\mu(X_0) = 0$, where $K_\mu \equiv \partial_\mu\theta$: $\mathbf{K} = (K_\mu) = (-\omega, \mathbf{k})$ is the wave (co)vector, with ω = frequency and $\mathbf{k}$ = "spatial" wave covector.

This leads to the *dispersion equation:*

$$\Pi_X(\mathbf{K}) \equiv a_0(X) + i\,a_1^{\mu}(X)K_\mu + i^2 a_2^{\mu\nu}(X)K_\mu K_\nu = 0.$$

Dispersion equation & wave equation

Clearly, the wave equation:

$$a_0(X)\psi + a_1^\mu(X)\partial_\mu\psi + a_2^{\mu\nu}(X)\partial_\mu\partial_\nu\psi = 0 \tag{1}$$

and the dispersion equation:

$$\Pi_X(\mathbf{K}) \equiv a_0(X) + i\, a_1^\mu(X)K_\mu + i^2 a_2^{\mu\nu}(X)K_\mu K_\nu = 0 \tag{2}$$

are in *one-to-one correspondence*.

Inverse correspondence (from (2) to (1)):

$$K_\mu \longrightarrow \partial_\mu/i, \quad (\mu = 0, ..., N \equiv \text{dim. of configur}^{\text{n}} \text{ space})$$

The *dispersion relation(s)*: $\omega = W(\mathbf{k}; X)$, fix the wave mode.
Obtained by solving $\Pi_X(\mathbf{K}) = 0$ for $\omega \equiv -K_0$.

The classical-quantum correspondence

Witham's theory of dispersive waves: still, propagation of $\mathbf{k}$ turns out to be ruled by a *Hamiltonian system*:

$$\frac{\mathrm{d}k_j}{\mathrm{d}t} = -\frac{\partial W}{\partial x^j},$$

$$\frac{\mathrm{d}x^j}{\mathrm{d}t} = \frac{\partial W}{\partial k_j} \quad (j = 1, ..., N \equiv \text{dimension of configuration space}).$$

Wave mechanics: a classical Hamiltonian H describes the skeleton of a wave pattern. Then, the wave eqn should give a dispersion W with the same Hamiltonian trajectories as H. Simplest way to do that: assume that H and W are proportional, $H = \hbar W$... *Leads first to* $E = \hbar\omega$, $\quad \mathbf{p} = \hbar\mathbf{k}$. *Then, substituting* $K_\mu \longrightarrow \partial_\mu / i$, *it leads to the correspondence between a classical Hamiltonian and a wave operator.*

(M.A.: *Nuovo Cim.* **B114**, 71, 1999.)

Algebraic vs polynomial Hamiltonians

For a non-relativistic particle, the Hamiltonian H is polynomial in the momentum $\mathbf{p}$ (at fixed X):

$$H(\mathbf{p}, X) = \frac{\mathbf{p}^2}{2m} + V(X), \tag{3}$$

and the correspondence leads to the Schrödinger equation. For a relativistic particle (say a free one), H is *algebraic* in $\mathbf{p}$ instead:

$$Q(E, \mathbf{p}) \equiv E^2 - \mathbf{p}^2 c^2 - m^2 c^4 = 0 \quad \text{if} \quad E = H(\mathbf{p}). \tag{4}$$

Applying the correspondence to (4), leads to the *Klein-Gordon eq.* The problem here is that (4) has also the solution $H' = -H$. Hence, the K-G eq. also will have *too much solutions.*

A variant derivation of the Dirac equation

Therefore, it is tempting to try a *factorization* of the dispersion equation associated with the algebraic relation (4):

$$\Pi(\mathbf{K}) \equiv (g^{\mu\nu}K_\mu K_\nu - m^2)1 = (\alpha + i\gamma^\mu K_\mu)(\beta + i\zeta^\nu K_\nu). \qquad (5)$$

Identifying coeffs. in (5) (with noncommutative algebra), and substituting $K_\mu \longrightarrow \partial_\mu / i$*, leads to the Dirac equation.*

This derivation works also with an electromagnetic field, and even also (as will be seen) in a static gravitational field.

(M.A.: *Found. Phys. Lett.* **19**, 225, 2006)

Transformation of Dirac wave function

One asks that, after linear coordinate changes L, the Dirac wave function ψ becomes

$$\psi'(X') = S.\psi(X), \qquad S = \mathsf{S}(L), \tag{6}$$

for some operator function S of L. It follows that Dirac eq. becomes

$$(i\gamma'^{\nu}\,\partial'_{\nu} - m)\psi' = 0, \quad \gamma'^{\nu} \equiv L^{\nu}_{\ \mu} S\gamma^{\mu} S^{-1}. \tag{7}$$

Standard statement: *Relativity asks that* $\gamma'^{\nu} = \gamma^{\nu}$ (whence the spinor representation). *But no!* Archetypically relativistic is the eq of motion for a particle with 4-velocity U^{μ} in e.m. field $F^{\mu}_{\ \nu}$:

$$m\frac{\mathrm{d}U^{\mu}}{\mathrm{d}s} = qF^{\mu}_{\ \nu}U^{\nu}, \qquad \text{or} \quad m\frac{\mathrm{d}U}{\mathrm{d}s} = qFU. \tag{8}$$

Here, matrix $F \equiv (F^{\mu}_{\ \nu})$ is *not* invariant: $F' = LFL^{-1}$.

4-vector transformation of Dirac wave function

The simplest possibility for S is the identity: $\mathsf{S}(L) = L$, thus the *4-vector transformation of the Dirac wave function:*

$$\psi'(X') = L.\psi(X), \qquad \text{or} \quad \psi'^{\mu} = L^{\mu}_{\nu}\psi^{\nu}. \tag{9}$$

Then, the Dirac matrices transform in the following way:

$$\gamma'^{\mu} \equiv L^{\mu}_{\nu} L \gamma^{\nu} L^{-1}, \tag{10}$$

which means that *the components* $(\gamma^{\mu})^{\rho}_{\nu}$ *form a* $\binom{2}{1}$ *tensor.*

- The anticommutation is preserved, $[\gamma'^{\mu}, \gamma'^{\nu}]_{+} = 2g'^{\mu\nu}\,1$.
- Direct physical consequences of the Dirac eq unchanged: the *equation*, hence its *solutions*, stay unchanged.

Gravitational Dirac equation from wave mechanics

For geodesic motion in a *static* metric, the particle's conserved energy E turns out to be a Hamiltonian (PIRT'98, gr-qc/0203104):

$$E \equiv \sqrt{g_{00}}\, mc^2 \gamma_v = H(\mathbf{p}, \mathbf{x}) = [g_{00}(h^{jk} p_j p_k c^2 + m^2 c^4)]^{1/2},$$

with $\boldsymbol{h} \equiv (h_{jk})$ the spatial metric in the static reference frame. With $E = \hbar\omega$, $\mathbf{p} = \hbar\mathbf{k}$, it leads to the same dispersion equation as in the flat case: $g^{\mu\nu} K_\mu K_\nu - m^2 = 0 \quad (\hbar = 1 = c)$. Hence, the same factorization leads to the same Dirac eq:

$(i\gamma^\mu\, \partial_\mu - m)\psi = 0$ (with "deformed" matrices, for $g^{\mu\nu} \neq \eta^{\mu\nu}$).

However, *our interpretation of the classical-quantum correspondence implies that it makes sense only in very special coordinate systems.* In the static case, these are such that

$$x^0 = at, \quad h_{jk,l}(\mathbf{x}) = 0, \quad j,k,l \in \{1,2,3\}.$$

The foregoing **"wave-mechanical" gravitational Dirac eq.** may be rewritten in the form below, covariant under coordinate changes which are *internal to the static reference frame:*

$$(i\gamma^{\nu}\Delta_{\nu} - m)\,\psi = 0, \tag{11}$$

with

$$(\Delta_{\nu}\psi)^{\mu} \equiv \psi^{\mu}_{,\nu} \qquad \text{if } \mu = 0 \text{ or } \nu = 0 \tag{12}$$

and

$$(\Delta_{\nu}\psi)^{\mu} \equiv \psi^{\mu}_{;\nu} \quad \text{(covariant derivative } \underline{\text{w.r. to } \boldsymbol{h}}\text{) otherwise.} \tag{13}$$

This is *definitely non-equivalent to the standard gravitational extension of the Dirac equation* – the latter being obtained by the *"covariantization"* of the Dirac eq. with spinor transform. Which is the more correct one? *Experiment*, in particular with **ultra-cold neutrons in gravity field (Nesvizhevsky et al., ILL)**, could tell us.

Hamiltonian operator for the Dirac equation in a static gravitational field

We are looking for *stationary solutions*

$$\psi(t, \mathbf{x}) = \phi(t)\, a(\mathbf{x}) \tag{14}$$

of the Dirac eq in a static metric. The *"wave-mechanical"* version of the latter eq. may be rewritten in Schrödinger form:

$$i\hbar \frac{\partial \psi}{\partial t} = \hbar c g_{00} (M \gamma^0 \psi - i \gamma^0 \gamma^j \Delta_j \psi) \equiv \mathrm{H}\psi, \qquad M \equiv \frac{mc}{\hbar}. \tag{15}$$

One shows that $\mathrm{H}\psi$ is a 4-vector under *spatial* coord. changes. And, there is just *one* invariant Hermitian product that appears naturally for 4-vector wave functions in a static metric.

Same job done also for the *standard* gravitational Dirac eq.

Weak-field limits vs. stationary Schrödinger eq. in the gravity potential U

Aim: to find corrections to the latter eq., and to compare these corrections as found with either the standard or the proposed version of the Dirac eq. in a static metric.

Framework: particle velocity is $O(\epsilon)$ and gravitational field strength U/c^2 is $O(\epsilon^2)$, with $\epsilon \ll 1$. Then, the stationary energy levels E should satisfy

$$E = \hbar\omega = mc^2 + O(\epsilon^2). \tag{16}$$

This automatically selects positive-energy solutions! Thus, a Foldy-Wouthuysen transform is unnecessary.

Weak-field limits vs. stationary Schrödinger eq. in the gravity potential U

The weak-field eqs have been calculated, first without checking hermiticity (!) The 2 put *different* corrections to Schrödinger eq in the gravity field.

It turns out that the Hamiltonian of the *standard* (Dirac-Fock-Weyl) equation *is* Hermitian for the relevant *invariant* scalar product. Corresponding weak-field eq.:

$$-\frac{\hbar^2}{2m}\Delta\varphi - mU\varphi + \frac{\hbar^2}{2mc^2}\left[\frac{3}{2}i\varepsilon_{jkl}\partial_j U\sigma^l\partial_k\varphi + \frac{1}{2}\partial_j U\partial_j\varphi - \frac{1}{2}\varphi\Delta U\right]$$
$$= (E - mc^2)\varphi + O(\epsilon^4).$$

In this eq., all terms are order ϵ^2, thus all have *the same order* in ϵ. *Corrections to Schrödinger eq. don't need to be negligible !!*

Summary and conclusions

- To write QM in a gravitational field: first interpret the classical-quantum correspondence.
- A variant derivation of the Dirac equation.
- *4-vector transformat$^{\mathrm{n}}$ of Dirac wave function.*
- *Dirac eq. with gravitation: wave mechanics clashes with equivalence principle.*
- Weak-field limits investigated to evaluate corrections to energy levels w.r.t. Schrödinger eq in gravity potential. These corrections will very likely differ for the 2 competing gravitational Dirac eqs. and *don´t need to be negligible.*