On the relation Hamiltonian – wave equation, and on non-spreading solutions of Schrödinger's equation

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Short title: **Relation Hamiltonian – wave equation and non-spreading wave packets**

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Abstract. According to Schrödinger's ideas, classical dynamics of point particles should correspond to the « geometrical optics » limit of a linear wave equation, in just the same way as ray optics is the limit of wave optics. It is shown that the « geometrical optics » analogy leads to the correspondence between a classical Hamiltonian H and a linear wave equation in a natural and general way. In particular, the correspondence is unambiguous also in the case where H contains mixed terms involving momentum and position. This is obtained through a theory of the dispersion relations, which leads to properly define the group velocity and to enlighten its role. Using this latter notion, it is shown that, for a quite general class of potentials, « momentum states » can be defined in a physically more satisfying way than by assuming plane waves. These momentum states are solutions of the time-dependent Schrödinger equation and their amplitude functions move rigidly on a well-defined trajectory. In the case of a spatially uniform force field, such momentum states must have a singularity and the trajectory is defined by Newton's second law in the given force field.

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1. Introduction

De Broglie found that the analogy between the variational principles which arise in geometrical optics and in Hamiltonian dynamics had its origin in a wave-particle duality valid for material particles as well as for electromagnetic radiation. Elaborating on de Broglie's ideas, Schrödinger proposed that classical dynamics of point particles should correspond to the « geometrical optics » approximation of a linear wave equation, in just the same way as ray optics is a limiting approximation of wave optics [1]. In quantum mechanics, the correspondence between a classical Hamiltonian and a quantum wave equation, first obtained by Schrödinger for the non-relativistic Hamiltonian of oneor several mass particles, is introduced in a general axiomatic way.

The first part of the present work is an attempt to understand this correspondence in the light of the modern theory of (classical) waves. More precisely, the correspondence between a linear wave equation and a classical Hamiltonian (algebraic in the momentum $\mathbf{p}$) will be analysed in the framework of the theory of dispersion relations in a linear medium. This theory is exposed *e.g.* by Whitham [2], but some definitions and results will have to be extended here. It will be shown that the above correspondence is naturally and most generally justified under the assumption that the classical Hamiltonian should describe the trajectories corresponding to the « geometrical optics » limit of the considered wave equation. Thus, the wave equation deduced from a classical Hamiltonian is *a priori* no more « quantum » than is the d'Alembert equation for the classical electromagnetic potential. This correspondence is *one-to-one* also in the general case where the Hamiltonian contains mixed terms involving momentum $\mathbf{p}$ and position $\mathbf{x}$ (and time). Contrary to the usual elementary presentation of this notion, the dispersion relation relevant to the definition of the group velocity does not depend on a particular wave packet, but instead on the considered *wave equation*. The crucial result which justifies the general correspondence Hamiltonian–wave equation is that, with any linear wave equation, one may associate a Hamiltonian dynamical system that rules the motion of the wave vector; in this system, the relevant velocity is the *group velocity* and the Hamiltonian is none

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other than the *dispersion* associated with the wave equation. In the « geometrical optics » limit, the wave function obeys simultaneously the dispersion equation and the wave equation, which is not true for general solutions of the wave equation

In the second part of this paper, the foregoing results will be used to reflect on what could be an appropriate definition of a « momentum state » for a quantum object, if one considers that a quantum object is really defined by a time-dependent wave function obeying a precise wave equation. From this viewpoint, an obvious sufficient condition allowing one to say that « the momentum is well-defined » is that (i) the group velocity should be spatially uniform. This property is true for a plane wave in the case of the usual Schrödinger equation, of course, but it is shared with a plenty of different wave functions. There are additional properties which should be satisfied by the wave function of a « momentum state » for a single non-relativistic particle: (ii) it should obey the time-dependent Schrödinger equation and, furthermore, (iii) it should be spatially limited in the minimum sense that the amplitude should vanish at infinity – a property obviously wrong for plane waves. It will be shown that, for a quite general *spatial* dependence of the potential, there indeed exist wave functions with just these properties. If the spatial dependence of the potential is unchanged with time, up to a translation and a linear function of the position, then the group velocity remains spatially uniform as the time goes. As a consequence, the amplitude function of any such wave function moves *rigidly*, thus, in particular, *it does not spread out at all,* and it follows a trajectory defined by integration of the group velocity. Non-spreading solutions of Schrödinger's equation have been found recently by Barashenkov & Rodrigues [3], but the amplitude functions of their solutions have a cylindrical symmetry, hence a such solution can hardly be interpreted as describing a moving quantum object localized in space. Moreover, in order to better understand the transition to classical mechanics, it is interesting to define particular momentum states, which may be called « classical momentum states », by imposing the additional condition that (iv) the wave function obeys the dispersion relation exactly. It will be shown that properties (i) to (iv), taken together, lead to the unavoidable conclusion that:

(a) classical momentum states can exist only in so far as the spatial variation of the force field is negligible, *i.e.*, the potential must be linear in **x**;

(b) in that case, they do exist, for whatever time-dependence of the force field;

(c) they *must* have a *singularity*, whose motion obeys *Newton's second law*.

In contrast, general momentum states do not necessarily have a singularity. We finish the paper by briefly commenting the meaning of these results in relation to the two different attempts, made by de Broglie and Schrödinger, to obtain a realistic interpretation of QM.

2. Dispersion relations and the correspondence Hamiltonian – wave equation

2.1 Dispersion equation and wave equation in a (classical) linear medium

As everyone knows, a general wave ψ may be defined as the product of a « smoothly varying » amplitude $A(t, \mathbf{x})$ and an oscillating part $\phi(\theta(t, \mathbf{x}))$, with ϕ a periodic scalar function of the dimensionless *real* variable θ, the latter or « phase » being itself a function of the time t and the position **x**. The « position » **x** belongs to an N–dimensional configuration space M, which may be the physical space ($N = 3$). In physics, ϕ is usually the complex exponential: $\phi(\theta) = \exp(i\theta)$ (or its real part, $\cos\theta$), but other « wave profiles » do also occur. The notion of a smoothly varying amplitude means that the wave structure may be recognized: to the very least, the relative amplitude variation $\delta A/A$ should not exceed the order of $\delta\theta / T$, with T the period of the wave profile ϕ. Now if we have a physical law for a given wave phenomenon, it will lead to a partial differential equation for the wave function ψ. Let us assume that ψ is a *scalar* and that this equation is *linear:*

$$(2.1) \qquad P\psi \equiv a_0(X)\ \psi + a_1{}^{\mu}(X)\ D_{\mu}\psi + \ldots + \sum_{\mu_0+\ldots+\mu_N=n} a_n{}^{\mu_0\ldots\mu_N}(X)\ (D_0)^{\mu_0}\ldots(D_N)^{\mu_N}\ \psi = 0,$$

where X, with coordinates x^{μ} ($0 \le \mu \le N$), is the relevant point of the product « time » × configuration space (extended configuration space $\mathbf{R}\times$M), and $D_{\mu}\psi = \psi_{,\,\mu}$ is the partial derivative. Quite often, the order of the equation is $n = 2$:

(2.2) $$P\psi \equiv a_0(X)\ \psi + a_1{}^{\mu}(X)\ \psi_{,\mu} + a_2{}^{\mu\nu}(X)\ \psi_{,\mu,\nu} = 0.$$

Given the phase function $\theta(X)$ of a general wave, one defines the wave covector $\mathbf{K} = \nabla\theta = (-\omega, \mathbf{k})$, *i.e.* $K_\mu = \theta_{,\mu}$. In the linear case, it is appropriate to consider a sinusoidal wave profile: $\phi(\theta) = \exp(i\theta)$, hence $\phi_{,\mu} = iK_\mu\phi$. Let us consider a wave function $\psi = A\exp(i\theta)$, with A constant, which is an « elementary wave » *at the point X considered*, *i.e.*, such that $K_{\mu,\nu} = 0$ at point X. The *necessary and sufficient* condition for such wave function to obey Eq. (2.2) at point X is:

(2.3) $$\Pi(X, \mathbf{K}) \equiv \Pi_X(\mathbf{K}) \equiv a_0(X) + ia_1{}^{\mu}(X)\ K_\mu - a_2{}^{\mu\nu}(X)\ K_\mu K_\nu = 0,$$

the obvious generalization to Eq. (2.1) being (under the condition that all derivatives of the K_μ 's, up to the order $n-1$, vanish at the point X considered):

(2.4) $$\Pi(X, \mathbf{K}) \equiv \Pi_X(\mathbf{K}) \equiv a_0(X) + ia_1{}^{\mu}(X)\ K_\mu + \ldots + i^n \sum_{\mu_0+\ldots+\mu_N=n} a_n{}^{\mu_0\ldots\mu_N}(X)\ K_0{}^{\mu_0}\ldots K_N{}^{\mu_N} = 0.$$

We shall name Eq. (2.3) (or (2.4)) the *dispersion equation* of Eq. (2.2) (or (2.1)). If one makes any coordinate change $x'^{\rho} = x'^{\rho}(x^\mu)$ in Eq. (2.2), the coefficient $a_0(X)$ is left unchanged and the $a_1{}^{\mu}(X)$ series transforms like a vector. However, the $a_2{}^{\mu\nu}(X)$ series transforms like a contravariant second-order tensor, if and only if one makes an « *infinitesimally linear* » coordinate change, *i.e.*, if $\partial^2 x'^{\rho}/\partial x^\mu \partial x^\nu = 0$ at the point considered. Hence, the left-hand side of the dispersion equation (2.3) is an invariant scalar only under infinitesimally linear coordinate transformations. Similarly, the condition $K_{\mu,\nu}(X) = 0$, used to derive (2.3) from (2.2), is covariant only under infinitesimally linear coordinate transformations. This means that, at each point X of the extended configuration space $\mathbf{R}\times M$, a linear group of privileged

coordinates must be available among the local coordinates valid in a neighborhood of X. Hence, $\mathbf{R}\times$M must be equipped with a more particular structure than just that of a differentiable manifold. At this point, the special role of the time coordinate does not appear compelling. A pseudo-Riemannian metric $\boldsymbol{\gamma}$ on $\mathbf{R}\times$M is hence enough: we then select the set of the locally geodesic coordinate systems (LGCS) at X for $\boldsymbol{\gamma}$ [$\gamma_{\mu\nu,\rho}(X) = 0$ for all μ, ν and ρ], two of which exchange indeed by an infinitesimally linear transformation. Thus, in order to evaluate the dispersion equation (2.3) at X, we can take any LGCS and we get the polynomial function Π_X of covector $\mathbf{K}$; this function does not depend on the LGCS. Hence, the function Π (of X and $\mathbf{K}$, thus a function defined on the « cotangent bundle », $T^*(\mathbf{R}\times M)$, to $\mathbf{R}\times$M) is well-defined. In the sequel we shall need the « projection time » t as a preferred time coordinate (up to a constant factor: $x^0 = \alpha t$).

Similarly, for an equation of order $n \geq 3$, the dispersion equation (2.4) is an invariant only under coordinate changes whose all derivatives, up to the order n, are zero at the point X considered. The condition that all derivatives of the K_μ 's, up to the order $n-1$, vanish at X, is also covariant under those changes only. And if we have a pseudo-Riemannian metric $\boldsymbol{\gamma}$, we might hope to define a privileged class of coordinates as the « n-LGCS systems », *i.e.* those in which all derivatives of the metric, up to the order $n-1$, are zero: this condition is stable under the changes just mentioned. However, already for $n = 3$, a « 3-LGCS system » exists only if the Riemann tensor vanishes at X. Hence, *the results presented in this paper are valid only for equations of order $n \leq 2$, or also for the case of a flat (configuration-)space–time metric* $\boldsymbol{\gamma}$ (in the latter case, if $n > 2$, one takes Galilean coordinates for $\boldsymbol{\gamma}$). This restriction has few practical consequences, obviously.

If the wave equation (2.1) has constant coefficients (a notion that is covariant only under truly linear coordinate changes, hence implies a vector space structure for $\mathbf{R}\times$M), it may indeed have solutions that are an « elementary wave » at *each point* X in an open domain of $\mathbf{R}\times$M (such solutions have a constant wave covector, hence are plane waves). But the

dispersion equation makes sense for a general linear wave equation. In order that the dispersion equation for the wave covector should have *real* solutions (which is the condition under which the general linear equation (2.1) really becomes a wave equation), one demands that its coefficients are real. Then, for a *real* wave equation, only odd derivatives, *or* only even derivatives, are allowed [2]. However, one may mix odd and even derivatives, provided one accepts complex coefficients in the wave equation – as for Schrödinger's equation.

Now we state the essential result of this Section: the correspondence between the linear operator P and the function Π is one-to-one, in other words one may *uniquely* pass from the wave equation to the dispersion equation *and conversely*. This is undoubtedly true, since the data of either P or Π is equivalent to the data of the set of the scalar functions a_0, $a_1{}^{\mu}$, $a_2{}^{\mu\nu}$, and so on, of the point X in $\mathbf{R}\times M$. The important point is that this result is true also in the general case with *variable* coefficients, in which a_0, $a_1{}^{\mu}$, $a_2{}^{\mu\nu}$, ..., do depend on X. In the case with constant coefficients, the inverse correspondence, from (2.4) to (2.1), amounts to the substitution

$$K_\mu \to \frac{1}{i} D_\mu . \tag{2.5}$$

This remains true in the case with variable coefficients, provided one orders each monomial in the dispersion equation as in (2.3) or (2.4), *i.e.*, « X before $\mathbf{K}$ ». This ordering is the natural ordering for a polynomial in $\mathbf{K}$ with coefficients that are functions of X. It corresponds to the ordering in Eq. (2.1) or (2.2), which also is the natural ordering for the *general* linear differential equation: the (variable) coefficient comes before the differentiation.

2.2 Group velocity and Hamiltonian motion

As mentioned above, we now need the preferred « projection » time t, with $X = (t, \mathbf{x})$, and we seek to compute the frequency $\omega = -K_0$ as a function of the « spatial » part $\mathbf{k}$ of $\mathbf{K}$: $\omega = \omega(k_1, ..., k_N) = \omega(\mathbf{k})$. When extracting the real roots of the polynomial equation $\Pi_X(\mathbf{K}) = 0$, considered as an equation for the unknown K_0 with the data $k_1, ..., k_N$, one may follow the different roots W_1,

..., W_n (at most n) as functions of X. It is assumed in the following that one such particular root has been identified, and we have thus:

(2.6) $$\omega = W(k_1, ..., k_N; X) = W(\mathbf{k}; X) = W(\mathbf{k}, \mathbf{x}, t),$$

which is called « the » *dispersion relation*, it being kept in mind that several different such relations may in general be extracted from the unique dispersion *equation* [2]. Of course, W, which will be called the *dispersion*, is in general not a polynomial function of $\mathbf{k}$ at fixed X. Now let us assume that some wave function of the general form

(2.7) $$\psi(t, \mathbf{x}) = A(t, \mathbf{x}) \exp(\mathrm{i}\, \theta(t, \mathbf{x})),$$

is such that the corresponding wave covector obeys exactly the *dispersion relation* (2.6) (we do *not* assume here that ψ is an exact solution of the *wave equation* (2.1), and indeed *this is in general incompatible with our assumption*). One defines the *group velocity* for this wave function as the « spatial » vector with components

(2.8) $$C^j = C^j(\mathbf{k}, \mathbf{x}, t) = \frac{\partial W}{\partial k_j}(\mathbf{k}, \mathbf{x}, t)$$

(Latin indices will be reserved for « spatial » components). This is the natural generalization [2] of the usual notion of group velocity in a uniform medium. From the definitions $K_\mu = \theta_{,\mu}$ and $\omega = -K_0$, and from the symmetry of the second derivatives of θ, one gets

(2.9) $$\frac{\partial \omega}{\partial x^j} + \frac{\partial k_j}{\partial t} = 0, \quad \frac{\partial k_i}{\partial x^j} = \frac{\partial k_j}{\partial x^i},$$

and since here $\omega(t, \mathbf{x}) = W(\mathbf{k}(t, \mathbf{x}), \mathbf{x}, t)$ by hypothese, Eq. $(2.9)_1$ may be rewritten as [2]

(2.10) $$\frac{\partial W}{\partial x^j} + C^i \frac{\partial k_j}{\partial x^i} + \frac{\partial k_j}{\partial t} = 0.$$

This is a hyperbolic equation for the (spatial) wave covector $\mathbf{k}$, showing that $\mathbf{k}$ propagates with the group velocity. It is immediate to verify that it can be put in the following characteristic form [2]:

$$\frac{\mathrm{d}k_j}{\mathrm{d}t} = -\frac{\partial W}{\partial x^j} \quad \text{on the curve} \quad \frac{\mathrm{d}x^j}{\mathrm{d}t} = C^j = \frac{\partial W}{\partial k_j}. \tag{2.11}$$

Thus, the motion of the wave vector in a general nonuniform medium is governed by a Hamiltonian dynamical system, the Hamiltonian being the dispersion W. To the author's knowledge, this crucial result has been overlooked in the literature on quantum mechanics.

We also point out that, to our knowledge, the present rigorous definition of the dispersion equation, and its rigorous correspondence with the wave equation, are new for the case of *variable* coefficients (i.e., a « non-uniform medium »): Whitham [2] gives the definition and the result only in the case of a uniform medium and says (on p. 382) that, « for a slightly non-uniform medium, it would appear reasonable to find the dispersion relation first for constant values of the parameters of the medium and then reinsert their dependence on $\mathbf{x}$, t. » Our method to extend the definitions and the result to the general case, *viz.* considering an « elementary wave *at point X* », leads us to discuss the covariance of the equations in a more general framework than that of absolute time plus Cartesian coordinates of the Euclidean space. Of course, the non-uniform case is essential for quantum mechanics since a non-uniform external field is always present in practice. It is important to note that (especially in the non-uniform case), a solution of the wave equation does not in general obey the dispersion equation, and *vice versa.*

2.3 The quantum correspondence and its (non-)ambiguity

Let us consider a classical system of point particles with « position » $\mathbf{x}$ (in the configuration space M), its dynamics being given by a Hamiltonian system with Hamiltonian H and conjugate momentum $\mathbf{p}$ (a « spatial » covector, though only under infinitesimally linear space coordinate changes, as one may check by studying the transformation of Eq. $(2.12)_1$):

(2.12) $$\frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H}{\partial x^j}, \quad \frac{\mathrm{d}x^j}{\mathrm{d}t} = \frac{\partial H}{\partial p_j}.$$

Following de Broglie and Schrödinger, let us imagine that those microscopic objects, which we initially described as point particles, actually have a spatial structure made of some (unknown) waves; and that this classical Hamiltonian dynamics only describes the « skeleton » of the wave motion, in *precisely the same way* as geometrical optics describes the trajectories of « light rays », which constitute the skeleton of the light wave propagation pattern. Thus, we expect that the dynamics (2.12) should be that approximation of the wave motion which becomes exact at the « nil wave length » limit where, in the neighborhood of any point X in the extended configuration space, the wave may be considered as a plane wave, that is

(2.13) $$A \approx \text{Const.} \quad \text{and} \quad \delta\theta \approx \mathbf{k}.\delta\mathbf{x} - \omega\,\delta t = K_\mu\,\delta x^\mu$$

in Eq. (2.7); of course, condition $(2.13)_2$ means that the higher-order derivatives of θ can be neglected: $K_{\mu,\,\nu} \approx 0$, and so on. We seek for the linear wave equation (2.1) for which the Hamiltonian dynamics would represent just that approximation. Assume that the wave function ψ, of the form (2.7), is the relevant *solution*, in a situation where the Hamiltonian dynamics is a good approximation, to that *exact* wave equation: in view of what we have just said, ψ will satisfy the approximate conditions (2.13). But the exact conditions $A = \text{Const}$, $K_{\mu,\,\nu} = 0$, and so on, are the conditions under which the substitution of (2.7) into (2.1) gives the dispersion equation (2.4). Hence, the wave covector $\mathbf{K}$ of an exact solution (2.7) to the wave equation (2.1) satisfies the dispersion equation (2.4) to an approximation which becomes exact in the « geometrical optics » limit where the dynamics (2.12) itself becomes exact.

Now, a *regular* solution of the dispersion equation (2.4) should correspond to a unique « branch », *i.e.*, it should satisfy one and only one among the different possible dispersion relations (2.6), and we assume that it is indeed the case. Therefore, in the « geometrical optics » limit, the spatial wave covector $\mathbf{k}$ [associated with a wave function (2.7) obeying the wave

equation (2.1)] obeys also the « Hamiltonian dynamics » (2.11). The only difference with a true Hamiltonian dynamics is that, in Eq. (2.11), a *continuous distribution* of Hamiltonian trajectories is involved. However, to account for the fact that (in its domain of relevance), the classical approximation (2.12) successfully considers point particles, we expect that, at least in some cases, the wave pattern is made of spatially concentrated « wave packets », in the exterior of which the amplitude is negligibly small. Obviously, *we are lead to admit that the Hamiltonian systems characterized respectively by the dispersion* W (Eq. (2.11)) *and by the classical Hamiltonian* H (Eq. (2.12)) *have exactly the same set of possible trajectories* $\mathbf{x} = \mathbf{x}(t)$.

This can be achieved by a *canonical* transformation $(\mathbf{x}, \mathbf{k}, t, W) \rightarrow (\mathbf{x}, \mathbf{p}, t, H)$ (thus with the « position » and time variables being left unchanged) only if $\mathbf{k} = \mathbf{p}$ and $W = H$. But a canonical transformation means that the trajectories in the extended phase space, $\mathbf{R}\times T^*M$, are the same: here we merely want that their *projections* in the extended configuration space $\mathbf{R}\times M$ are the same. If we consider the Lagrangians $\Lambda(\mathbf{x}, \dot{\mathbf{x}}, t)$ and $L(\mathbf{x}, \dot{\mathbf{x}}, t)$ respectively associated with W and H by the Legendre transformation, the condition is that the extremals of the respective action integrals must be the same. The obvious, simplest way to ensure that this is true, is to impose that Λ and L are equal up to a multiplicative constant. We denote this constant by $\hbar$ ... Thus, we are lead to admit that $L = \hbar \Lambda$, whence by Legendre transform:

(2.14) $$H = \hbar W, \qquad \text{or} \qquad E = \hbar \omega,$$

and for the canonical conjugate momenta:

(2.15) $$\frac{\partial L}{\partial \mathbf{x}} = \hbar \frac{\partial \Lambda}{\partial \mathbf{x}}, \qquad \text{or} \qquad \mathbf{p} = \hbar \mathbf{k}.$$

The *a priori* interpretation of the correspondence (2.14) and (2.15) is a *formal* one: it is the correspondence between the Hamiltonian systems (2.11) and (2.12) – that govern respectively (i) the bicharacteristics of the propagation equation (2.10) for a wave covector obeying the dispersion relation (2.6), and (ii) the trajectories of a classical system – under the condition that

the « geometrical optics » limit of the wave equation (2.1) gives the same trajectories as the classical system. Moreover, let us recall that the wave equation can be *uniquely* deduced from the dispersion *equation* (2.4) (see the end of § 2.1). However, the correspondence (2.14) gives us the dispersion *relation* (2.6). But the dispersion relation is just one root of the polynomial dispersion equation (2.4), when the latter is seen as an equation for $\omega = -K_0$.

Thus, if we are given a classical Hamiltonian H and if we seek for the associated « quantum » wave equation, we first (trivially) deduce the dispersion W by the correspondence (2.14) and (2.15), *i.e.*

(2.16) $$W(\mathbf{k}, \mathbf{x}, t) = [H(\hbar\,\mathbf{k}, \mathbf{x}, t)]/\ \hbar\,.$$

Then: *either* (i) H is a polynomial in $\mathbf{p}$ (at fixed X), such as, for instance,

(2.17) $$H(\mathbf{p}, \mathbf{x}, t) = \frac{\mathbf{p}^2}{2m} + V(t, \mathbf{x}).$$

In this case, the data of H (or W) already represents the dispersion equation: more exactly, it is the simplest (lowest degree) dispersion equation that has W as one possible dispersion. Therefore, the correspondence (2.5) gives unambiguously the wave equation. This is also true if the coefficients of the polynomial *depend* on $\mathbf{x}$ and t. Note that the composition of the two correspondences $(E, \mathbf{p}) \rightarrow (\omega, \mathbf{k})$ [Eqs. (2.14) and (2.15)] and (2.5) gives the « quantum correspondence » :

(2.18) $$E \rightarrow +\mathrm{i}\hbar\frac{\partial}{\partial t}, \quad p_j \rightarrow -\mathrm{i}\hbar\frac{\partial}{\partial x^j}.$$

From Eq. (2.17), one thus gets Schrödinger's equation:

(2.19) $$\mathrm{H}\psi \equiv -\frac{\hbar^2}{2m}\Delta\psi + V\psi = \mathrm{i}\hbar\frac{\partial\psi}{\partial t}.$$

Or (ii) H is not a polynomial in $\mathbf{p}$, but some polynomial function of H is itself a polynomial in $\mathbf{p}$. In that case, we take the dispersion equation as that polynomial function Π which has the lowest degree, which is necessarily unique up to a constant factor, and we thus obtain uniquely the linear wave equation with the lowest possible order. In practice, the Hamiltonian H will

then appear precisely as involved in such polynomial function: this is the case for the relativistic Hamiltonian of, for instance, a free particle, which appears in the equation

(2.20) $$[H(\mathbf{p})]^2 - \mathbf{p}^2 c^2 - m^2 c^4 = 0.$$

For the latter dispersion equation (up to the correspondence (2.14) and (2.15)), the wave equation obtained by the correspondence (2.5) is, of course, the Klein-Gordon equation, with the d'Alembert equation as the case $m = 0$. The latter case shows that the approach based on the « geometrical optics » limit does work in the genuine case! It also shows that *the quantum correspondence gives merely the wave equation for anyone scalar component, and tells us nothing about the scalar or tensor character of the physically relevant wave field.*

To be complete, there is still the case that (iii) H is not an algebraic function of $\mathbf{p}$, that is, no polynomial function of H becomes a polynomial in $\mathbf{p}$. This means simply that no linear wave equation can be associated with the given classical Hamiltonian by this correspondence.

2.4 Comments

The usual correspondence (2.18), between a classical Hamiltonian and a « quantum » wave equation, may thus be arrived at by a seemingly new method, based on the dispersion relation in a linear medium. The interest one may find to this method is that it gives a rather rigorous basis to the essence of wave mechanics: the idea that classical dynamics of point particles should correspond to the « geometrical optics » limit of a linear wave equation. It has been shown that this idea leads, modulo the study of dispersion relations, to the correspondence (2.18) in a natural and general way. The method is not limited to the case of a single particle: it works for any Hamiltonian system. Moreover, the method also indicates how to make the correspondence unambiguous in the case where the classical Hamiltonian contains mixed terms involving both the momentum $\mathbf{p}$ and the (configuration-)space–time position $X = (t, \mathbf{x})$. The rule to obtain the wave operator unambiguously is simple: just put the function of X as a

multiplying coefficient before the monomial in $\mathbf{p}$ – as the dispersion equation has to be a polynomial in $\mathbf{p}$ at fixed X. This is important in the case with a gravitational field, for a curved space-time means that, for example, the kinetic energy becomes a sum of such mixed terms (since it involves the metric tensor, that depends on X).

Thus, a classical Hamiltonian (even in an implicit form such as Eq. (2.20)) gives unambiguously a linear wave equation for the more complex wave structure supposed to underly the approximate classical behaviour of a system of particles (the latter corresponding to the nil wave length limit). However, if one admits that it is a wave structure which is the fundamental behaviour, then one must expect, of course, that in some cases no classical Hamiltonian will be relevant, *i.e.*, one must expect that the simple correspondence (2.18) is not always a sufficient tool to obtain a correct wave equation – an example is Dirac's equation.

3. Momentum states and non-dispersive solutions of Schrödinger's equation

3.1 Definition of momentum states: a generalization of plane waves

In QM, it is admitted that a (single) quantum particle has a well-defined momentum $\mathbf{p}$, if and only if its wave function (in the physical space, which coincides, in the case of a single particle, with the configuration space) is that of a plane wave. Moreover, only the spatial dependence of the wave function is envisaged, so that the « wave function » (at a given time) is $A \exp(i\, \mathbf{k}.\mathbf{x})$. Thus, the first wave function encountered in QM has constant modulus and, in particular, it is not square-integrable, although for the case of one quantum « particle » obeying Schrödinger's equation, this is a necessary requirement. Actually, it is easy to show that a plane wave cannot obey the time-dependent Schrödinger equation unless the potential V is a constant number (thus unless the « particle » is in a free motion); this will be proved in passing below. One may ask if the wave function of a *moving* particle could be defined as a function of *time* also, and if its spatial extension could be limited in some sense.

Thus, we consider a wave function of the general form (2.7), and we ask under which particular conditions it could be taken to represent a quantum object having, at any given time

t, a well-defined momentum $\mathbf{p}(t)$. The velocity of the particle is related to the group velocity of the associated wave (Eq. (2.11)), indeed it must coincide with the latter in the « geometrical optics » limit since, in this limit, Eqs. (2.11) and (2.12) must give the same trajectories. Therefore, an obvious sufficient requirement is that (i) *the group velocity* $\mathbf{C}$ *must be a function of* t *only*, thus a given vector at a given time, which will represent the velocity of the particle. However, $\mathbf{C}$ depends on the dispersion W (Eq. (2.8)). The latter is found by the correspondence (2.16), in the particular case that the quantum wave equation can be deduced from a classical Hamiltonian by the correspondence (2.18). In the general case, the dispersion equation is found directly from the wave equation by the transition from Eq. (2.1) to Eq. (2.4), thus by the substitution $\mathrm{D}_\mu \to \mathrm{i}\, K_\mu$, provided the wave equation can be put in a form where it is the same for each component of the wave function; then, several dispersions (whence several $\mathbf{C}$ vectors) may be defined by extraction (see § (2.2)). In any case, the definition of $\mathbf{C}$ depends on the wave equation. Hence, the natural condition (i) implies that *the very definition of a « momentum state » should depend on the quantum wave equation*. Let us assume that this is the one-particle Schrödinger equation (2.19), associated with the non-relativistic Hamiltonian (2.17). The group velocity is then

(3.1) $$\mathbf{C} = \hbar\mathbf{k}/m,$$

so that, in the case of the usual Schrödinger equation, condition (i) is equivalent to the requirement that (i)' *the spatial wave covector* $\mathbf{k}$ *is a function of* t *only* [1]. Thus $\nabla_{\mathbf{x}}\theta = \mathbf{k}(t)$, which is equivalent to say that the phase θ has the form

(3.2) $$\theta(t, \mathbf{x}) = \mathbf{k}(t).\mathbf{x} - f(t).$$

Note that, under condition (i)', the frequency is not necessarily uniform, indeed

[1] This is not true in general, *e.g.* it is wrong for the (scalar) Schrödinger equation in the presence of a magnetic vector potential $\mathbf{A}$: in that case, one finds $\mathbf{C} = [\hbar\mathbf{k} - (q/c)\mathbf{A}]/m$ (consistently with the fact that, for a classical particle governed by the relevant Hamiltonian, $\mathbf{P} - (q/c)\mathbf{A} = m\dot{\mathbf{x}}$ is the « true » momentum, where $\mathbf{P}$ is the

(3.3) $$\omega = f'(t) - \mathbf{k}'(t).\mathbf{x}.$$

Obviously, condition (3.2) is fulfilled by a plenty of possible wave functions, since no restriction is imposed on the amplitude A. But since we recognized that our definition of a momentum state depends on the wave equation, we have one more reason to impose on the wave function ψ the condition that (ii) *ψ should obey the wave equation* – which is anyway a necessary requirement [2]. Furthermore, we assume from now on that the amplitude A, as well as the phase θ, is *real*. This is the obvious necessary condition under which the writing of the complex function ψ in the form (2.7) is (practically) unambiguous: otherwise, the relation between ψ and its phase θ would be totally ambiguous. (The restriction that A also should be real does not play a role in the study of dispersion relations, since one then considers constant amplitudes.) After an easy algebra, the Schrödinger equation (2.19) for the wave function (2.7), with the phase (3.2), is then found equivalent to the two real equations:

(3.4) $$\mathrm{H}A \equiv -\frac{\hbar^2}{2m}\Delta A + VA = \left(\hbar\omega - \frac{\mathbf{p}^2}{2m}\right)A, \qquad (\mathbf{p}(t) \equiv \hbar\mathbf{k}(t)),$$

(3.5) $$\frac{\mathrm{d}A}{\mathrm{d}t} \equiv \frac{\partial A}{\partial t} + \mathbf{C}.\nabla A = 0.$$

Equation (3.5) shows that the amplitude A undergoes a mere translation with the uniform group velocity $\mathbf{C}(t)$. In precise terms: for any trajectory $t \mapsto \mathbf{x}(t) = \chi_t(\mathbf{x}_0)$ (with $\mathbf{x}_0$ the position at $t = 0$) such that, any time t, $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{C}(t)$, then A is a constant:

(3.6) $$A\left(t, \mathbf{x}_0 + \int_0^t \mathbf{C}(s)\mathrm{d}s\right) = A(0, \mathbf{x}_0).$$

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canonically conjugate momentum, to which applies the relation $\mathbf{P} = \hbar\mathbf{k}$). Hence, $\mathbf{k} = \mathbf{k}(t)$ is then *incompatible* with $\mathbf{C} = \mathbf{C}(t)$ unless $\mathbf{A} = \mathbf{A}(t)$ (*i.e.,* no magnetic field).

[2] In the case where a magnetic field is present, the expression of the group velocity involves $\mathbf{A}$ that depends on the gauge condition (note 1). Therefore, one is really enforced to impose the wave equation: if one changes the gauge condition, the wave function, hence also the wave covector $\mathbf{k}$, will then transform according to a simple rule [4], and it is easy to verify that the group velocity is then independent of the gauge.

Since these trajectories are all identical but for a constant space translation, one may say that the quantum object follows a well-defined trajectory, provided both equations (3.4) and (3.5) are satisfied in the whole space and for a non-zero interval of time. Let us ask whether this is possible.

3.2 Existence of momentum states for a general spatial dependence of the potential

In addition to the amplitude function $A(t, \mathbf{x})$, the unknowns in Eqs. (3.4) and (3.5) are the wave vector $\mathbf{k}(t)$ determining the group velocity $\mathbf{C}(t)$, Eq. (3.1), and the scalar function $f(t)$ which, together with $\mathbf{k}(t)$, determines the phase θ, Eq. (3.2). Due to this unusual set of unknowns, the discussion is a bit subtle. In order to establish for which kind of potentials $V(t, \mathbf{x})$ one may find solutions to these equations, we consider the initial value problem: *we assume the initial data* $A_0(\mathbf{x}) = A(0, \mathbf{x})$ (for all $\mathbf{x}$), $\mathbf{k}(0)$ *and* $f(0)$. If one knows the dependence $\mathbf{k} = \mathbf{k}(t)$, Eq. (3.5) is completely solved by Eq. (3.6) and the initial data $A_0(\mathbf{x})$ for all $\mathbf{x}$.

Now Eq. (3.4) at $t = 0$ may be seen as restricting the possible initial data for a given initial potential $V_0(\mathbf{x}) = V(0, \mathbf{x})$:

$$(3.7) \qquad H_1 A_0 \equiv -\frac{\hbar^2}{2m}\Delta A_0 + \left(V_0 + \hbar \mathbf{k}'(0).\mathbf{x}\right) A_0 = \mu A_0, \qquad \mu \equiv \hbar f'(0) - \frac{\hbar^2 \mathbf{k}(0)^2}{2m}.$$

Hence, an easy way to find a licit initial data A_0 for a quite general initial potential V_0, is to assume the additional data $\mathbf{k}'(0)$ (amounting to the data of the initial acceleration!) and to determine A_0 and μ (hence $f'(0)$) as an eigenfunction and the corresponding eigenvalue for the problem (3.7). This is simply an eigenvalue problem for the modified Hamiltonian H_1, deduced from the true Hamiltonian operator H by adding the linear term $\hbar \mathbf{k}'(0).\mathbf{x}$ to V_0, thus by adding a uniform force field, just as in the Stark effect. If V_0 is a « typical » potential of QM, it is hence clear that there should be solutions to this problem, and this is indeed the case for a nil potential as well as for the Coulomb potential [5, pp. 440-442]. In order to get a well-defined problem with well-defined solutions, one still needs to precise the boundary conditions for A_0 in the eigenvalue problem $(3.7)_1$. As is usual in QM, we may content ourselves by imposing the

condition that (iii) A_0 *vanishes at infinity* ($A_0(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$). Note that, conversely, if one starts from a licit initial data A_0 and μ, Eq. $(3.7)_1$ determines the initial acceleration of the particle, $\mathbf{a}_0 = \hbar\mathbf{k}'(0)/m$. In other words, there is no undetermination for $\mathbf{a}_0$, but the mere fact that, in a given arbitrary potential, the initial amplitude of the wave function cannot be arbitrary for a momentum state. Also note that one still may consider $(3.7)_1$ as defining a particular potential, the now arbitrary initial amplitude function A_0 being given.

Having shown that the restriction (3.7) imposed to the initial data A_0, furthermore assigned to vanish at infinity, may be satisfied for a quite general initial potential V_0, we now have to ask if Eqs. (3.4) and (3.6) may hold as the time goes. Thus, the amplitude $A(t, \mathbf{x})$ is known from Eq. (3.6) if the time evolution $\mathbf{C}(t)$, or equivalently $\mathbf{k}(t)$, is known. Inserting (3.6) into (3.4), we get

$$V(t,\mathbf{x}) = \hbar\left(-\mathbf{k}'(t).\mathbf{x} + f'(t)\right) + \frac{\hbar^2}{2m}\left[-\mathbf{k}(t)^2 + \frac{\Delta A_0}{A_0}\left(\mathbf{x} - \int_0^t \mathbf{C}(s)\,\mathrm{d}s\right)\right], \tag{3.8}$$

whence

$$V(t,\mathbf{x}) = V_0\left(\mathbf{x} - \int_0^t \mathbf{C}(s)\,\mathrm{d}s\right) + \hbar\left[\left(\mathbf{k}'(0) - \mathbf{k}'(t)\right).\mathbf{x} + f'(t) - f'(0) + \frac{\hbar}{2m}\left(\mathbf{k}(0)^2 - \mathbf{k}(t)^2\right)\right]. \tag{3.9}$$

Therefore, *up to its translation that must follow the motion of the quantum object, the initial potential must change only by a time-dependent linear function,*

$$V(t,\mathbf{x}) = V_0\left(\mathbf{x} - \int_0^t \mathbf{C}(s)\,\mathrm{d}s\right) + \mathbf{u}(t).\mathbf{x} + U(t). \tag{3.10}$$

Conversely, if this is the case, then, by rewriting (3.10) in the form (3.9), the data of this linear function (together with the initial data $A_0(\mathbf{x})$ which fixes $\mathbf{k}'(0)$, see above) determines uniquely $\mathbf{k}'(t)$ for all t; hence, accounting for the initial data $\mathbf{k}(0)$, it determines $\mathbf{k}(t)$ and $\mathbf{C}(t)$ for all t. Thus, it also determines $f'(0) - f'(t)$ for all t. We see that $f'(0)$ must be added to the initial data, although, for a given potential, the data $f'(0)$ is subject to the constraint that μ (Eq. $(3.7)_2$) is an eigenvalue of the problem $(3.7)_1$. Thus, if the time evolution of the potential obeys Eq. (3.9), we define $A(t, \mathbf{x})$ by Eq. (3.6). It is then almost obvious, and easy to check, that Eq. (3.4) continues to be verified for $t > 0$.

In summary, there indeed exist momentum states, defined as wave functions that (i) have a uniform group velocity, (ii) obey the time-dependent Schrödinger equation and (iii)

have a vanishing amplitude at infinity, *provided that* the initial potential changes only through a uniform translation determined by a time-dependent linear function of the position, Eq. (3.10). The arbitrary time-dependence of that linear function determines the time evolution of the (spatially uniform) momentum and that of the frequency (which is spatially non-uniform, except for the case where the momentum is conserved, *i.e.* the case $\mathbf{u}(t) \equiv 0$ in Eq. (3.10), plus $\mathbf{k}'(0)=0$). The amplitude function of these momentum states moves rigidly with the motion. The evolution equation of the momentum is got by applying the gradient operator to Eq. (3.8):

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \equiv \hbar\mathbf{k}'(t) = -\nabla V(t,\mathbf{x}) + \frac{\hbar^2}{2m}\nabla\left(\frac{\Delta A_0}{A_0}\right)\left(\mathbf{x} - \int_0^t \mathbf{c}(s)\,\mathrm{d}s\right), \tag{3.11}$$

it gives, by construction, a uniform evolution if Eq. (3.9) is satisfied. One recognizes the second term in the r.h.s. of Eq. (3.11) as a particular case of the well-known « quantum potential » (see *e.g.* Vigier [6]).

Thus, *there exists a very general class of non-spreading solutions of Schrödinger's equation, each of which represents a « localized object » in the sense that the amplitude vanishes at infinity, and with a well-defined momentum.* This is in striking contradiction with the standard presentation of quantum mechanics, in particular with the usual exegesis of the uncertainty principle, though it does not actually contradict either the latter principle or QM in general (see the discussion in § 3.4). Note that condition (3.10), imposed to the time-dependence of the potential, is less artificial than it may appear at first sight. First, it is satisfied by any time-independent potential (in this case, our result is not very interesting since any *real* solution of the Schrödinger equation for *stationary states* yields a wave function with similar properties, once it is multiplied by the sinusoidal time-dependence in order to get a solution of the time-dependent Schrödinger equation). More importantly, condition (3.10) is satisfied by any quantum object (1) assumed to remain at rest with respect to a moving object (2) with which it interacts, provided the interaction potential (2→1) is independent of the motion of the system (1)+(2) (which may be approximately true), and provided the action on (1) of the exterior of the system (1)+(2) is a spatially uniform force field. An obvious subcase is the case

where (1) is actually "alone", it is hence the potential corresponding to a *spatially uniform force field*. It is worth to study that case separately.

3.3 « Classical » momentum states

In view of Eq. (3.4), we may wish to impose to a momentum state the additional requirement that (iv) *the dispersion relation is satisfied:*

$$\hbar\omega - \frac{\mathbf{p}^2}{2m} - V = 0. \tag{3.12}$$

One might guess that those particular momentum states may have something to do with the classical limit of the wave equation because, as we have seen, the very construction of wave mechanics is based on the assumption according to which the classical limit corresponds to a situation (the geometrical optics) in which the wave equation and the dispersion equation can be verified simultaneously. However, since the geometrical optics is a *limit*, one might expect that, in situations relevant to this limit, these two equations cannot be verified *exactly*. This is not the case, as we shall now see. The requirement (3.12) implies that Eq. (3.4) is equivalent to the Laplace equation:

$$\Delta A = 0. \tag{3.13}$$

Moreover, Eq. (3.12), in which we recall that $\mathbf{p} = \mathbf{p}(t) \equiv \hbar\mathbf{k}(t)$, implies also that V is a linear function of $\mathbf{x}$: since ω is given by Eq. (3.3), we get indeed from (3.12):

$$V(t, \mathbf{x}) = \hbar f'(t) - \hbar\mathbf{k}'(t).\mathbf{x} - \frac{\hbar^2}{2m}\mathbf{k}^2. \tag{3.14}$$

Conversely, if we know that $\Delta A = 0$, then, unless $A(t, \mathbf{x}) = 0$, Eq. (3.4) is equivalent to Eq. (3.12) and thus implies Eq. (3.14). In particular, a plane wave (A = Const, $\mathbf{k}$ = Const, ω = Const) can be a solution of the Schrödinger equation only if the potential V is a constant, as announced. This result, plus the fact that a plane wave has a constant modulus over the space, seems to imply that a plane wave cannot represent a correct description of the objective state of a single quantum object having a well-defined momentum. The more general result expressed by Eq. (3.14) is that the dispersion relation cannot be verified for a momentum state which is

an exact solution of the Schrödinger equation, unless the potential V is linear in space, *i.e.*, unless the force field $\mathbf{F}$ is uniform. In contrast to the case of a general momentum state (Eq. (3.11)), the evolution of $\mathbf{p}$ is purely classical if the potential has this form (3.14):

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \equiv \hbar \frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t} = -\nabla V \equiv \mathbf{F}(t). \tag{3.15}$$

Hence, momentum states satisfying the additional condition (iv) will be called « classical momentum states ». We thus find that, for the usual Schrödinger equation, *a classical momentum state can exist only over space and time intervals such that the traversed force field may be considered as spatially uniform.*

Thus, considering a uniform force field $\mathbf{F}(t)$, let us study such a classical momentum state precisely. Given the initial momentum $\mathbf{p}_0$, its time evolution is completely determined by *Newton's second law* (3.15). Due to Eq. (3.1) with $\mathbf{k} = \mathbf{k}(t)$, the momentum may really be considered as uniformly distributed in the wave (as for any momentum state). The dispersion relation (3.12), *i.e.*, « $E = \hbar\omega$ », determines the frequency ω. Note that it depends on space and time. The spatial variation of A is determined by the Laplace equation (3.13). Here, we wish that (a) A is defined *in the whole space* (except perhaps at a *point* singularity) and harmonic, and (b) A tends towards zero as $r \equiv |\mathbf{x}| \to \infty$. First, these conditions imply that there is a ball $|\mathbf{x}| \leq r_1$ such that A is not bounded in this ball, since otherwise it would be harmonic except perhaps at one point, and bounded in the whole space, hence [7, p. 305] it would be harmonic in the whole space; being harmonic and bounded in the whole space, it would be a constant, hence zero by condition (b). Hence, A is not harmonic at one point of the ball $|\mathbf{x}| \leq r_1$ (since otherwise it would be continuous, hence bounded, in this ball), which we choose as the origin. Thus, *A has a point singularity* at $\mathbf{x} = 0$. Now if we impose the subsidiary condition that A is not more singular than $1/r$, *i.e.*, rA is bounded as $r \to 0$, then we get by the same theorem [7, p. 305] that there is a constant R such that $A - R/r$ is a harmonic function ϕ on the whole space. But ϕ is bounded on the whole space since it is bounded in every bounded domain and since, as A and $1/r$, it tends towards zero as $r \to \infty$. Hence, ϕ is a constant α, which must be zero. Thus,

conditions (a) *and* (b) *imply that* $A = R/r$ with R an arbitrary constant and $r \equiv |\mathbf{x}|$ after the appropriate choice of the origin $\mathbf{x} = 0$, *unless* one would accept that A be more singular than $1/r$. Condition (b) gives $A \approx R/r$ as $r \to \infty$, *independently* of the latter condition [7, p. 314]. Unfortunately, R/r is not square-integrable. Note that, if we replace condition (b) by the less severe condition that (b)' *A is bounded as* $r \to \infty$, just the same arguments lead to $A = \alpha + R/r$. Now let us consider, for R and $\alpha > 0$, the function

$$(3.16) \qquad A_{R\alpha}(\mathbf{x}) = -\alpha + R/r \ \text{ if } 0 < r \leq R/\alpha, \qquad A_{R\alpha}(\mathbf{x}) = 0 \ \text{ if } r \geq R/\alpha.$$

From a physical point of view, this continuous function is undistinguishable from R/r if α is small enough, and it has a finite square integral. Of course, A does not satisfy $\Delta A = 0$ on the sphere $r = R/\alpha$, but A vanishes from that sphere and, in traversing the sphere, its derivatives undergo an arbitrarily small discontinuity if α is small enough. We note finally that different amplitudes functions A may be found if one accepts that A has, for instance, a one-dimensional singularity (such as a circle). However, as long as one demands that A vanish at infinity, a singularity is unavoidable since A must be a harmonic function.[3]

3.4 Comments on the physical interpretation of the momentum states

Thus, *in a uniform force field* $-\nabla V = \mathbf{F}(t)$, *the Schrödinger equation admits the wave functions*

$$(3.17) \qquad \psi(t, \mathbf{x}) = A_{R\alpha}[\mathbf{x} - \mathbf{a}(t)] \exp\{ i [\mathbf{k}(t).\mathbf{x} - f(t)] \}, \qquad d\mathbf{a}/dt = \hbar\mathbf{k}(t)/m,$$

as square-integrable classical momentum states, i.e., the group velocity (3.1) is spatially uniform, the Schrödinger equation *and* the dispersion equation (3.12) are satisfied. The wave

[3] Probably, A must also have a singularity if one demands instead that A vanish outside a bounded open domain D and be harmonic in D, but perhaps not on its boundary: *e.g.* this is true if D is the ball $|\mathbf{x}| < R'$, as for $A_{R\alpha}$ [7, p. 390].

covector $\mathbf{k}(t)$ and the function $f(t)$ are determined (up to their arbitrary initial value) by the potential $V(t, \mathbf{x})$; in particular, the momentum $\mathbf{p}(t)=\hbar\mathbf{k}(t)$ evolves according to Newton's second law (3.15). The *arbitrary* numbers R and α determine the *constant* « spatial extension » of a such wave function, which may be taken to be $R' = R/\alpha$. As outlined above, α should be considered as a small « cut-off ». In replacing $A_{R\alpha}$ by R/r, one gets classical momentum states that satisfy Schrödinger's equation everywhere except at the singularity and that vanish at spatial infinity, though without having a finite square integral. The fact that the evolution of the momentum is governed by Newton's second law seems to make the transition from classical mechanics to wave mechanics quite transparent: classical mechanics would envisage as a point particle what is in fact a spatially extended wave. And since a « classical momentum state » remains so only in a uniform force field, classical mechanics would be valid only in the case that the force fields vary slowly in space.

Moreover, we have now definitely proved by a class of analytical examples that, *contrary to the standard presentation of QM, a « quantum particle »* (as defined by a square-integrable wave function obeying Schrödinger's equation) *may have a finite and constant spatial extension* ***and*** *a well-defined momentum*. Furthermore, *the amplitude of the wave function increases as* $1/r$ *towards a point singularity which moves according to Newton's second law.*[4] These *classical* momentum states look like an exact realization of de Broglie's ideas according to which a quantum object would involve both a « real » wave and a singularity whose motion is « guided » by the wave (see *e.g.* Vigier [6]). Note, however, that in the case of a *general* momentum state, the amplitude function A_0 is defined by a rather generic eigenvalue problem of quantum mechanics (Eq. $(3.7)_1$) and so, in general, should not exhibit any singularity, even with the additional condition that A_0 vanishes at infinity. In our opinion, if one wishes to interpret the present results in the framework of physical realism (and if, moreover, one accepts that quantum mechanics is an exact description), they are more likely to

[4]If one accepts a one-dimensional singularity, the behaviour near this singularity will have to be determined, *e.g.* it is not guaranteed in advance that a square-integrable function may be obtained. Anyhow, the amplitude will also undergo a pure translation with the uniform group velocity obeying Newton's second law (Eqs. (3.1, 3.6, 3.15)).

sustain Schrödinger's conception according to which the « matter waves » are the true reality: for Schrödinger, « ... a particle is not a permanent entity », and « ...[one should] hold on to the wave aspect throughout the whole process. » [8]

A question arises then: how is this result compatible with Heisenberg's uncertainty principle? The answer is, obviously, that the latter applies (as a consequence of the QM postulates regarding the « collapse of the wave function ») to the outputs of *measurements* of momentum and position. Whereas the present result applies to the objective description of a « quantum particle » (wave function) in a spatially uniform force field, thus excluding a necessarily perturbating action of measurement. This result is hence compatible with the axioms of QM if they are taken in the operational sense, and it has no immediate consequence on the possible predictions that one can make in QM. Moreover, we emphasize that what is called here a « momentum state » is *not* an eigenfunction of the « momentum operator » in the direction of the current wave vector. We also note that, since the spatial extension of a « classical momentum state » is *arbitrary*, one still may say in a restricted sense that « the particle has not a well-defined position ». (This is no surprise if one considers that there is indeed no point particle but an extended wave.) Another one might consider, on the contrary, that « the position of the particle » is perfectly defined at any time in the case of a classical momentum state, as the position $\mathbf{a}(t)$ of the singularity. However, for a general momentum state and, even more generally, for a generic solution of the time-dependent Schrödinger equation, there is no singularity which might indicate the position of a point particle.

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