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ORIGINAL ARTICLE

Probabilistic analysis of a pull-out test

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Abstract This paper presents a sensitivity analysis of the pull-out strength of reinforcement embedded in concrete. Considering both European and French design codes, this failure strength depends on the variability of uncertain parameters such as Young's modulus of concrete and yield stresses of materials (concrete and steel); moreover, two failure modes can be observed in the studied experimental test. A methodology allowing the characterization of the sensitivity of the pull-out strength to these uncertain parameters is derived. These parameters are modeled by Lognormal random variables. Results show the evolution of the pull-out strength for different anchorage lengths. Probability density functions of the random variable modeling the failure strength are computed using probabilistic methods. A finite element model is also built to quantify uncertainties concerning failure modes, computing 95% confidence intervals.

Keywords Pull-out test · Failure modes · Stochastic finite element method · Monte Carlo simulation · Probabilistic sensitivity analysis · Nonlinear damage mechanics

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1 Introduction

The pull-out strength of reinforcement embedded in concrete depends mainly on material and geometrical characteristics of the assembly. Both French [1] and European [2] design codes in reinforced concrete construction consider that this strength depends on the variability of the Young's modulus of concrete and yield stresses of materials (concrete and steel). Different failure modes also depend on these parameters. Design codes [1, 2] take into account uncertainties on material characteristics using safety factors and characteristic values. This semi-probabilistic approach uses 5% fractile of the uncertain parameters as input data for failure strength calculation. But uncertainties on modes failure should be quantified too.

This study aims at taking into account uncertainties on materials and on failure modes in the analysis of a pull-out test.

Thus this work completes others studies characterising the mechanical failure: pull-out strength [3], crack propagation [3, 4], and influence of anchor shape [5]. A finite element (FE) model is often used herein due to the complexity of this problem, as reflected by the nonlinearity of constitutive laws [6–9] and issues dealing with modelling of the steel-concrete interface [7, 10–14]. Nevertheless, only one study of a pull-out test by means of both a non linear damage model and a probabilistic approach was found [8]. Spatial variability of concrete is taken into account, but only one failure mode is considered and no probability density function of the peak load has been evaluated.

By taking into account the statistical variability of uncertain mechanical parameters, stochastic finite element methods (SFEM) [15, 16] have been developed over the past 30 years and provide an alternative to the well-known Monte Carlo simulations [17]. Featuring a greatly reduced computation time, these approaches may be applied to complex FE models. A so-called "non-intrusive" group of SFEM refers to methods that do not modify the actual FE model, and these would include response surface methods. This category of methods has inspired research work using Hermite polynomials [18, 19]. Other efforts [20, 21] have shown that a Lagrange polynomial basis may be more precise and less time-consuming in seeking to obtain statistical moments (mean, variance, etc.) and probability density functions (PDF). This "Lagrange method" has recently been applied to a steel connection with material and geometric nonlinearities [22] and entailed characterizing some of the mechanical response parameters.

This paper serves as complementary research on both a non linear modelling of pull-out tests with a basis in probabilistic tools. Two probabilistic methods will be used: common Monte Carlo simulations; and the Lagrange method, which for the first time will be applied to a composite connection at failure, for the purpose of evaluating the first-order moments and probability density functions (PDF) of failure strength. The evolution in failure strength will be characterised for various anchoring lengths, in considering the variability of input mechanical parameters, such as Young's modulus of concrete and yield stresses of both concrete and steel.

Probabilistic methods for sensitivity analyses are introduced first along with the FE model of the described pull-out test. A deterministic evolution of failure strength is then computed, with two failure modes being examined; numerical results agree with experimental findings. Next, Monte Carlo simulations and Lagrange method are applied to the FE model, while material behaviour remains elastic. Results from both methods are in good agreement with one another, and the Lagrange method is eventually used to study failure modes. The variability in failure strength for various anchoring lengths is characterised using coefficients of variation and a 95% confidence interval. The paper will conclude with comparisons involving experimental results and design codes (French BAEL91 [1] and European EC2 [2]).

2 Failure strength obtained by means of a pull-out test

2.1 Presentation of the test

Experimental pull-out tests studied below, concern two different configurations where the variable parameter is the anchorage length. A steel reinforcement is embedded in a concrete sample. These pull-out tests are realized with 8 and 32 cm of embedding length. Material parameters are summarised in Table 1 and correspond to those identified by some available experimental tests (concrete compressive tests or steel tensile tests) or given by French design codes [1]. Two failure modes can be observed. In the first case of 8 cm of embedding, the steel is sliding out of the concrete (mode 1, Fig. 1). On the contrary, the 32 cm of embedding steel reach the maximal strength and breaks (mode 2, Fig. 2). The steel reinforced bar is pulled out applying a vertical force. Ten pull-out tests are available, experimental means and coefficients of variation of failure strength are given in Table 2 for both of these modes.

2.2 Failure modes from design codes

Considering both failure modes 1 and 2, French design code for reinforced concrete structures [1]

Table 1 Mechanical parameters of the finite element model

Parameter	Mean value	Description
E_b	30 GPa	Young's modulus of concrete
ν_b	0.2	Poisson's ratio of concrete
ρ_b	2.300 kg/m^3	Concrete density
f_{c28}	30 MPa	Concrete compressive yield strength
E_s	210 GPa	Young's modulus of steel
ν_s	0.3	Poisson's ratio of steel
ρ_s	7.850 kg/m^3	Steel density
f_y	500 MPa	Steel yield strength
H_s	21 GPa	Steel hardening modulus ($E_s \times 10\%$)

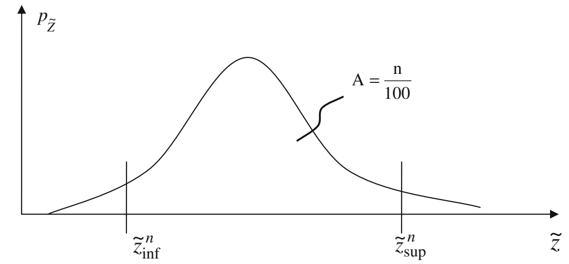


Fig. 1 Evolution of the probability density function of random variable and n% confidence interval $[\tilde{z}^n_{inf}; \tilde{z}^n_{inf}]$

stipulates respective values of failure strength $F = F_1$ or $F = F_2$.

$$F = \min\begin{cases} F_1 = \pi \times L_S \times \phi \times (0.6 + (0.06 \times \mathrm{f_{c28}})) \\ F_2 = \pi \times \phi^2 \times \mathrm{f_y}/4 \end{cases} \quad (1)$$

where ϕ is the diameter of the reinforced steel bar, $\mathrm{f_{c28}}$ and $\mathrm{f_y}$ the material yield stresses (respectively concrete steel). If the anchorage length L_s is greater than 10 cm, the European design code [2] gives similar values. From these simple formulas, it seems useful to study the sensitivity of F to the variability of $\mathrm{f_{c28}}$ and $\mathrm{f_y}$.

2.3 Presentation of the finite element model

A finite element (FE) model is built from available pull-out tests, in order to illustrate the following probabilistic methodology. In this work, the strategy is thus to combine this model to a probabilistic approach. It is why a compromise between refinement of the model and its ability to reproduce experimental tests has to be found. In other words, the FE model has to be as simple as possible, in order to allow a statistical treatment.

A two-dimensional axisymmetric model will be considered stemming from the problem geometry (see Figs. 2, 3). The computation is performed in large displacements (an actualized Lagrangian).

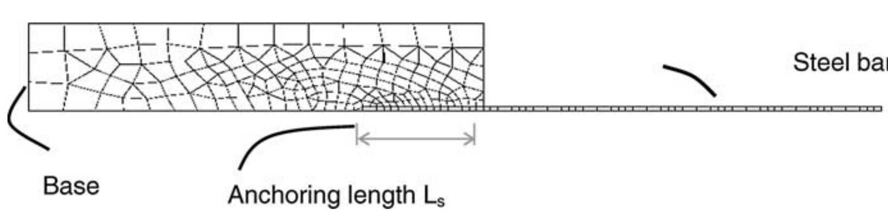
Table 2 Experimental results: means and standard deviations of the r.v. modelling the failure strength for anchoring lengths $L_s = 8$ cm and $L_s = 32$ cm

	F ($L_s = 8$ cm)	F ($L_s = 32$ cm)
Mean	22 kN	33 kN
Standard deviation	2 kN	1 kN
Coefficient of variation	7%	3%

Boundary conditions are imposed longitudinally at the base of the concrete specimen and then radially along the axis of symmetry. A displacement is prescribed on the free edge of the steel bar. Various analyses based on non linear modelling of concrete have shown their ability to model the pull-out test [6–9]. In this work, the concrete constitutive model is based on an elastic law with damage (Mazars' model [23]). The parameters characterising this law have been chosen in order to reproduce model mechanical characteristics of concrete given in Table 2. The steel bar constitutive model is elasto-plastic with hardening. A simplified model without any bond stress versus the slip relation at the steel-concrete interface is thus obtained. Indeed, because of the use of reinforced steel bars, damage due to micro-cracking of concrete is not taken into account, that has already been deemed equivalent to a perfect bond law model [7]. Eventually, the refinement of the mesh has been chosen as simple as possible, in order to achieve agreement with experimental results and to allow a statistical treatment.

With this objective, numerical criteria denoted D_i and ϵ_s are proposed: $D_i = 0$ represents a structurally-sound concrete, while $D_i = 1$ depicts a damaged concrete; ϵ_s is a deformation limit set for steel equal to 10‰ [1]. Figures 4–6 show respectively the evolution in maximum steel strain ϵ_s, evolution in steel-concrete interface damage D_i, and evolution in failure strength F for various anchoring lengths ($2 \leq L_s \leq 32$ cm). These evolution patterns can be broken down into three parts:

Fig. 2 Finite element model mesh of the steel-concrete half-connection ($\sim 10^2$–10^3 elements)



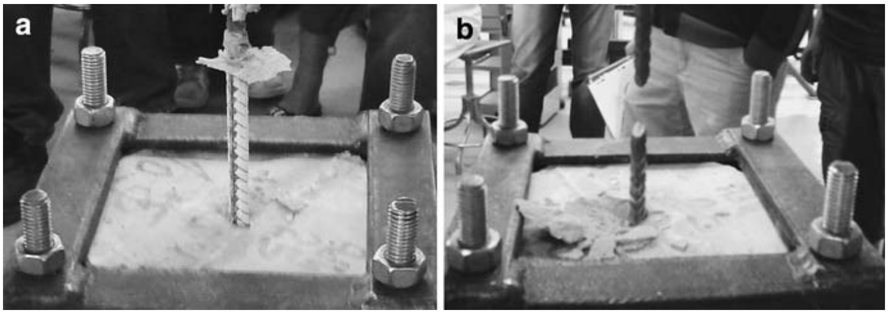


Fig. 3 Failure modes—Mode 1: Bond failure at the steel/concrete interface (**a**), Mode 2: Steel bar failure (**b**)

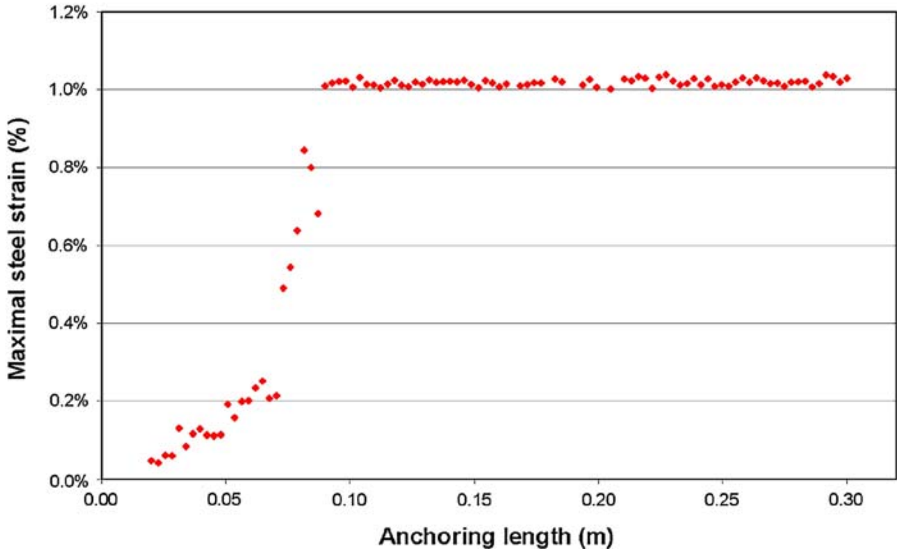


Fig. 4 Evolution in maximum steel strain ϵ_s for various anchoring lengths ($2 \leq L_s \leq 32$ cm)

- if $L_s < 9$ cm, D_i values nearly equal 1 and failure occurs for small steel strain ϵ_s values (i.e. less than 0.8%). Failure strength F increases linearly with anchoring length L_s (see Fig. 5). This part characterises the concrete damage and bond failure;
- if $L_s > 15$ cm, steel strain ϵ_s values nearly equal 1% and D_i is decreasing. Failure strength F is constant and equal to the steel strength (see Fig. 4). This part characterises the steel "failure" (plastic yielding); and
- if 9 cm $< L_s <$ 15 cm, failure occurs for constant values of failure strength F, which is equal to the steel strength (see Fig. 6). This part therefore would seem to correspond with failure mode 2 (plastic yielding). Yet uncertainty is still obviously present on the failure mode, due to D_i values nearly equalling 1.

In order to characterise this uncertainty, we will attempt in the following discussion to quantify the sensitivity of failure strength evolution to the variability of three input parameters: the failure stress of concrete f_{c28} and the yield stress of steel f_y and also the Young's modulus of concrete E_b.

3 Sensitivity analysis of the pull-out test

3.1 Probabilistic sensitivity approach

Let's consider the uncertain parameters of a mechanical system, as modelled by random input variables (r.v.) $\mathbf{Y} = \{Y_1,\ldots,Y_E\}$ with known probability distributions. The mechanical system is called f, such that $\mathbf{Z} = f(\mathbf{Y})$ is a vector output r.v. $\mathbf{Z} = \{Z_1,\ldots,Z_S\}$ to be characterised. For the sake of simplicity, we will focus on the special case of scalar input and output variables, i.e. $\mathbf{Y} = Y_1 = Y$ and $\mathbf{Z} = Z_1 = Z$.

If the mechanical function is simple (analytical function or linear finite element model), Monte Carlo

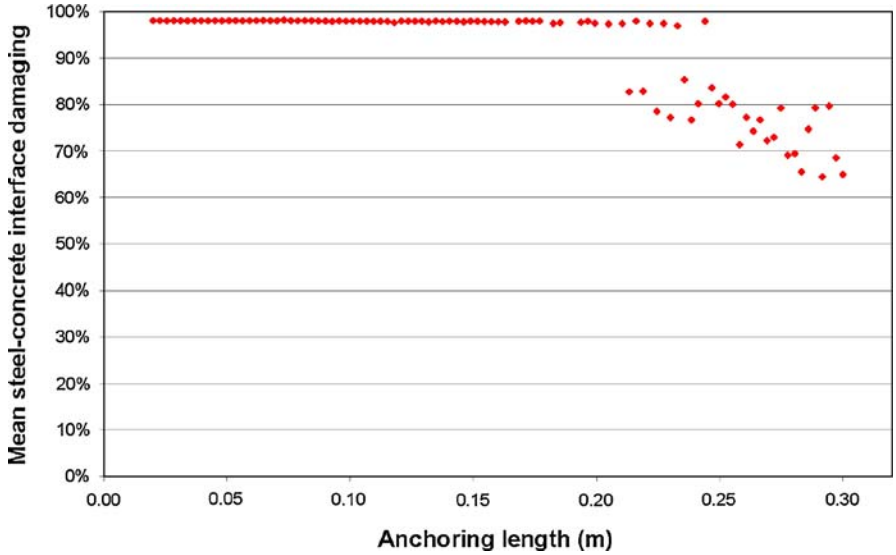


Fig. 5 Evolution in steel-concrete interface damage D_i for various anchoring lengths ($2 \leq L_s \leq 32$ cm)

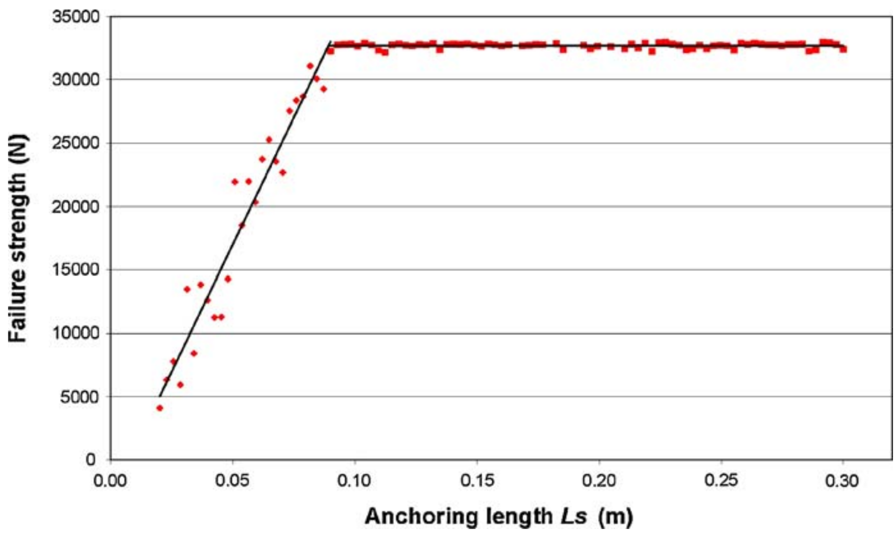


Fig. 6 Evolution in failure strength F for various anchoring lengths ($2 \leq L_s \leq 32$ cm)

methods can be used. These methods [17] are based on the same principle, which consists in selecting N values for input r.v. Y and then independently computing for each value y_i the mechanical response $z_i = f(y_i)$ of the system. But if f represents a numerical model, even time consuming, some alternatives like stochastic finite element methods (SFEM) are preferred. In this work, A probabilistic method based on Lagrange polynomials is chosen.

Statistical moments (mean, variance), probability density function (PDF) and $n\%$ confidence interval I_n are estimated from an approximation $\tilde{Z}$ of the output random variable Z. The N points required for this Lagrange method approximation are called "integration points" (see Appendix).

3.2 Application to the composite connection (elastic behaviour)

A scalar lognormal input r.v. Y is considered and serves to model variability in the Young's modulus of concrete E_b, with a mean $\mu = 3.10^{10}$ Pa and a coefficient of variation $C_v = 10\%$ (i.e. the standard deviation over mean). The output r.v. Z modelling the variability of maximum strength F_{max} is obtained as a 1-μm displacement and applied to the free edge of the steel bar.

We will now focus on comparing Monte Carlo simulations and the Lagrange method.

3.2.1 *Monte Carlo simulations*

Different simulations have been performed for both modes and for an increasing number of samples ($10^3 < K < 10^5$), with each sample corresponding to a mechanical FE computation. Because of the high computational cost associated with this simulation, a maximum of 10^5 samples have been computed.

Let's now consider the 10^5 sample simulation estimations as the target results: the means of Z for both mode 1 ($L_s = 8$ cm) and mode 2 ($L_s = 32$ cm) are approximated by the estimations denoted $\hat{\mu}_Z^1$, equal to 35.0906 N, and $\hat{\mu}_Z^2$, 35.5984 N, respectively; moreover, the standard deviations of Z are approximated by the estimations denoted $\hat{\sigma}_Z^1$, equal to 0.1741 N, and $\hat{\sigma}_Z^2$, 0.1345 N, respectively.

For other quantities of samples ($K < 10^5$), relative errors (in percentage terms) with respect to the above target results may be identified: Tables 2, 3 shows the numerical convergence of these relative errors, for the two failure modes, as the number of samples K increases. Given this convergence, the target results are assumed to be sufficiently accurate.

3.2.2 *Comparison with the Lagrange method*

Statistical moments and PDF approximations will now be compared with target results for the failure modes. The Lagrange method approximations are obtained for various integration points ($3 \leq N \leq 7$). Relative errors on the expected values lie below 0.01%, regardless of the number of integration points N for both modes. As for the standard deviation, errors tend to decrease as the number of integration points N increases, while remaining below 4% (mode 1) and 2% (mode 2).

The PDF of response Z can be studied by examining Fig. 7, which shows the estimated PDFs of the r.v. Z. These PDFs have been obtained by Monte Carlo simulations of the approximated response $\tilde{Z}$ (see Appendix, Eq. (7)) and are denoted $p_{\tilde{Z}}$. Lagrange method approximations $\tilde{Z}$ are derived for various integration points ($3 \leq N \leq 7$). In Fig. 7, PDF curves are shown only for $N = 3$ and $N = 7$, in mode 1, with the other curves ($N = 4, 5, 6$, mode 2) being almost superimposed. In comparing these approximated PDFs with the PDF estimated by direct Monte Carlo simulation in the deterministic FE model (target simulation, $K = 10^5$), a good level of agreement seems to be observed between the target PDF and the approximated ones.

3.2.3 *Conclusion*

A number $N = 4$ integration points is considered sufficient to obtain good results on PDF and statistical moments, in comparison with a Monte Carlo method using 10^5 calls. The Monte Carlo method is not feasible for failure analysis due to time-consuming computations inherent in the pull-out FE model (from a few minutes to several hours). A 4-point Lagrange method will therefore be used in the following for the pull-out test failure analysis.

The validity of the SFEM for n-dimensional cases was demonstrated in [18], with n limited to 4 or 5 for practical reasons. [18] showed that the validity in a one-dimensional case can be extended to the n-dimensional case while random variables remain independent, as it will be the case in the following.

Table 3 Relative errors on the mean and standard deviation target estimations ($\hat{\mu}_Z^1$; $\hat{\mu}_Z^2$; $\hat{\sigma}_Z^1$; $\hat{\sigma}_Z^2$), obtained for $K = 10^5$ Monte Carlo simulations (elastic behaviour, failure modes 1 and 2)

	Relative mean errors ($\times 10^{-3}$%)		Relative standard deviation errors (%)	
K	Mode 1	Mode 2	Mode 1	Mode 2
10^3	10.0	7.5	3.2	3.2
5×10^3	5.1	3.8	1.3	1.3
10^4	2.4	1.9	1.1	1.1
5×10^4	0.6	0.4	0.2	0.2
10^5	$\hat{\mu}_Z^1 = 35.0906$ N	$\hat{\mu}_Z^2 = 35.5984$ N	$\hat{\sigma}_Z^1 = 0.1741$ N	$\hat{\sigma}_Z^2 = 0.1345$ N

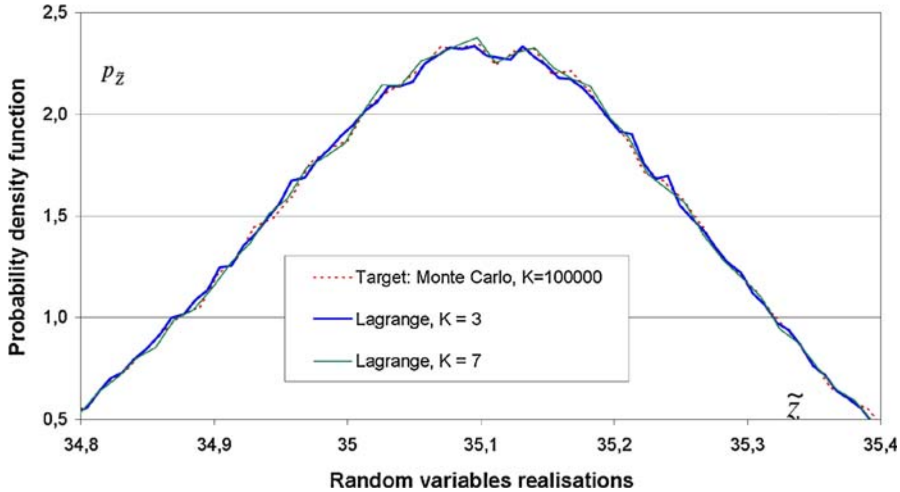


Fig. 7 Evolution in the probability density function $p_{\tilde{Z}}$ of the r.v. $\tilde{Z}$, with both Monte Carlo simulation (10^5 FE model runs) and Lagrange method (3 and 7 runs), mode 1, elastic behaviour

3.3 Application to the failure analysis

The first set of failure computations is conducted with one or two input r.v. modelling the variability of mechanical parameters, such as Young's modulus of concrete E_b, failure stress of concrete f_{c28} and yield stress of steel f_y. The output r.v. serves to model the failure strength F. Let Cv(f_{c28}), Cv(f_y) and Cv(F) denote the coefficients of variation of r.v.s. modelling the variabilities of f_{c28}, f_y and F, respectively.

Figure 8 depicts the evolution of Cv(F) for different values of Cv(f_{c28}) and Cv(f_y); this figure shows the sensitivity of F to the variability of r.v. modelling f_y in mode 2. A similar figure has been generated, revealing the sensitivity of F to the variability of r.v. modelling f_{c28} in mode 1.

The same analysis has then been performed for L_s ranging between 2 and 32 cm, in the aim of characterising failure modes. Three analyses were carried out, one for each uncertain parameter E_b, f_{c28}

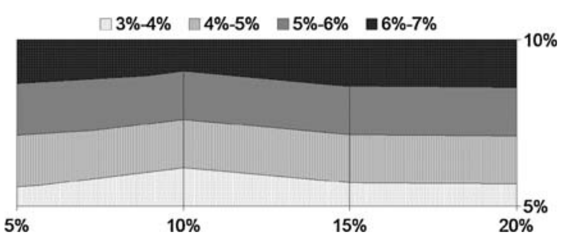


Fig. 8 Evolution in the coefficient of variation $Cv(F)$ of failure strength F, with increasing coefficients of variation for material yield stresses (concrete: f_{c28} and steel: f_y), failure mode 2, anchoring length $L_s = 32$ cm

and f_y, considering arbitrarily coefficients of variation Cv(E_b), Cv(f_{c28}) and Cv(f_y) equal to 10%. Figure 9 presents the failure strength F evolution for various anchoring lengths L_s. For each value of L_s, a dispersion interval has been computed that corresponds to the maximum variability of the three parameters with a $\pm$ 1 standard deviation, which once again leads to three areas:

- The first, in which F increases linearly with anchoring length L_s, corresponds to concrete damage and bond failure; this area is associated with small values of L_s (<10 cm) and dispersion intervals here are due solely to E_b and f_{c28} variabilities.
- The second area, in which F remains constant and equal to steel strength, corresponds to plastic yielding of the steel bar; this area is associated with high values of L_s, namely $L_s > 13.5$ cm, and dispersion intervals here are due solely to f_y variability.
- The intermediate area ($10 < L_s < 13.5$ cm) reflects an uncertainty on the failure mode resulting from variability of all three input parameters, corresponding to the ± 1 standard deviation intervals; this area would tend to increase for higher dispersion intervals.

This study remains indicative as long as a confidence interval has not been associated with these variation intervals. This condition requires knowing the PDF of the mechanical response Z at each computation point, a step that can be achieved by

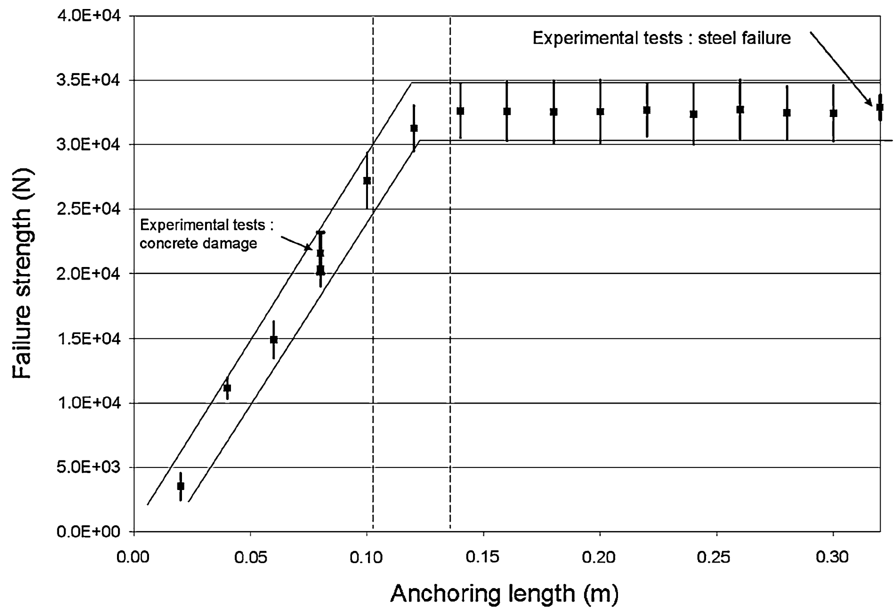


Fig. 9 Evolution in failure strength for various anchoring lengths ($2 \leq L_s \leq 32$ cm), as obtained by finite element computation—Sensitivity to mechanical parameters: Young's modulus of concrete E_b, material yield stresses (concrete: f_{c28} and steel: f_y)—A 1-standard deviation interval is associated with each mean failure strength

applying a Monte Carlo method on the analytical approximation $\tilde{Z}$ of the response Z given by the Lagrange method (see Appendix, Eq. (7)).

Figure 10 shows failure strength F evolutions for each anchoring length. The failure strength values F, as stipulated by design codes [1] and [2], are also provided along with all mean SFEM computations. These values reach those of the design code, which is necessary yet not enough to assess whether or not these codes are safe: confidence intervals would also be required. For this reason, PDFs $p_{\tilde{Z}}$ of the r.v. $\tilde{Z}$ are performed. For anchoring lengths $L_s = 8$, 10 cm, the PDF $p_{\tilde{Z}}$ is obtained by considering the uncertain parameter f_{c28}. For anchoring length $L_s = 24$ cm, the PDF $p_{\tilde{Z}}$ is obtained by considering the uncertain parameter f_y. The PDFs are truncated only on the 95% confidence intervals. It is shown herein that the confidence interval of these design codes exceeds

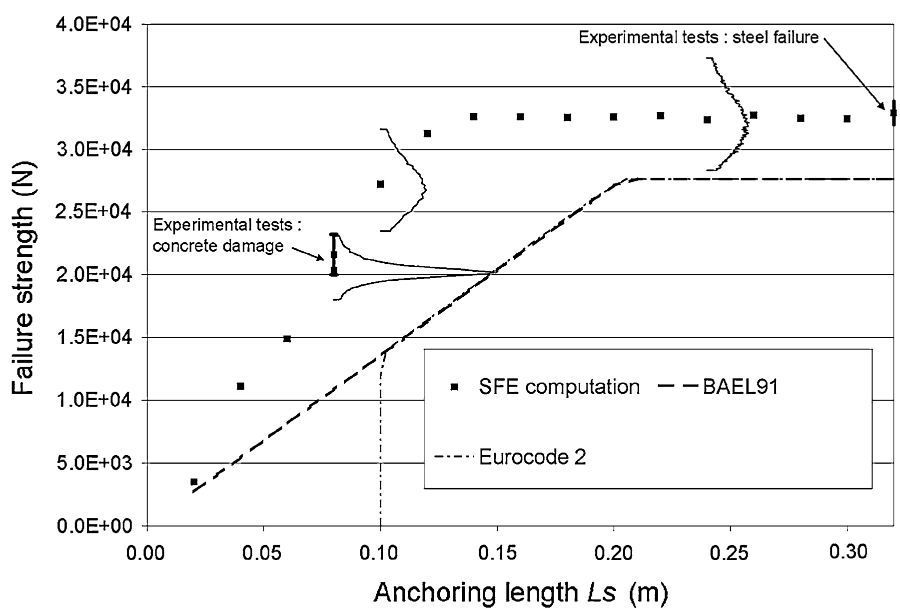


Fig. 10 Probability density functions $p_{\tilde{Z}}$ of the r.v. $\tilde{Z}$ for anchoring lengths $L_s = 8$, 10 cm (uncertain parameter: f_{c28}) and $L_s = 24$ cm (uncertain parameter: f_y)—PDFs are only truncated on the 95% confidence intervals—Failure strength limits extracted from design codes (Eurocode 2 [2] and BAEL91 [1])

95%. Such a probabilistic analysis therefore seems to indicate differing safety levels between failure modes 1 and 2. The apparently greater safety margin for concrete failure has however been justified by more uncertain characteristics of the concrete and steel-concrete interface. A reliability analysis and refined FE model would certainly yield a critical approach towards the design codes, and ongoing research is currently addressing this issue.

4 Conclusion

Uncertainties on the parameters of a system can lead to the use of probabilistic methods as a means of evaluating their effect on system responses. Such methods however prove to be time-consuming. One solution to this issue has been obtained by employing stochastic finite element methods (SFEM). Unlike some time-consuming methods, such as Monte Carlo simulations, SFEM may be feasible for conducting failure computations. This approach has been illustrated here by setting up a recent SFEM method based on Lagrange polynomials. A probabilistic study of the pull-out test of a steel bar anchored into concrete is indeed original and offers a complementary analysis to other deterministic studies of this mechanically nonlinear problem (once again using a recent SFEM). Various sensitivity indicators have been presented: means, standard deviations, coefficients of variation, and probability density functions, for the different failure modes. This sensitivity analysis has been conducted with regard to failure strength versus variability of this system's mechanical parameters: Young's modulus of concrete, yield stresses of both materials. The FE model has been built to be in agreement with failure modes observed during experimental tests. The variation in this strength versus anchoring length has also been computed, and a dispersion interval associated with this evolution allows characterising the uncertainty on failure strength and modes. The SFEM approximation of the mechanical response constitutes an analytical estimation, on which a Monte Carlo method has been applied. An approximation of the PDF of the r.v. modelling failure strength has thus been computed, and this has confirmed the potential of associating a confidence interval with failure strength variability. Moreover, extending such a sensitivity analysis, in association with a reliability analysis, would lead to a critical analysis of the design codes.

Appendix: probabilistic methods for sensitivity analysis

Monte Carlo simulations

Different Monte Carlo methods [17] are based on the same principle, which consists of selecting K values for input r.v. Y and then independently computing for each value y_i the mechanical response $z_i = f(y_i)$ of the system. It is possible to estimate the statistical moments of output r.v. Z, whose mean μ_Z and variance σ_Z^2 are approximated such that:

$$\mu_Z \approx \tilde{\mu}_Z = \frac{1}{K}\sum_{i=1}^{K} z_i \quad (2)$$

$$\sigma_Z^2 \approx \tilde{\sigma}_Z^2 = \frac{1}{K}\sum_{i=1}^{K} z_i^2 - \tilde{\mu}_Z^2 \quad (3)$$

where σ_Z is the standard deviation of Z.

Expressions (2) and (3) can be generalised to E input r.v. and S output r.v., and the approximations improve as K increases. Practically speaking however, the number of mechanical computations K should range from 10^4 to 10^7 in order to produce accurate approximations of statistical moments or probability density functions (PDF). This slow convergence rate prevents the use of Monte Carlo simulations for nonlinear computing that lasts more than a few hours.

To prevent this situation from arising, stochastic finite element methods (SFEM) have been developed over the past 30 years [15, 16]. SFEM allow approximating statistical moments and PDF, as well as sensitivity indices of output r.v. with a reduced number of mechanical model iterations. One recent model will be considered herein: the Lagrange method [20, 21].

Lagrange method

Let N be a nonzero integer and $(x_i)_{1\leq i\leq N}$ a set of N real numbers (collocation points). The basic idea here is to approximate the mechanical response f, which is a real function of real value x, by projecting it onto the truncated basis $\{L_i\}_{i=1\ldots N}$ of Lagrange polynomials

$$f(x) \approx \tilde{f}(x) = \sum_{i=1}^{N} \alpha_i \cdot \prod_{\substack{k=1 \\ k \neq i}}^{N} \frac{x - x_k}{x_i - x_k} = \sum_{i=1}^{N} \alpha_i \cdot L_i(x) \tag{4}$$

where α_i is the weight associated with polynomial L_i such as

$$\forall i \in \{1\,;N\} \quad \alpha_i = f(x_i) \tag{5}$$

By substituting (5) into (4), the approximation $\tilde{f}$ of f becomes:

$$\tilde{f}(x) = \sum_{i=1}^{N} f(x_i) \cdot L_i(x) \tag{6}$$

Now, let g be the composite function $f \circ T$ of the mechanical response f binding Z to a continuous r.v. Y with known PDF, and the function T binding Y with a standard r.v. (i.e. with a mean of 0 and standard deviation of 1) (s.r.v.) X (Gaussian normalisation) [16].

Combining the expression of $\tilde{f}$ obtained in (6), the r.v. Z is approximated by r.v. $\tilde{Z}$, such that:

$$\tilde{Z} = \tilde{g}(X) = \sum_{i=1}^{N} g(x_i) \cdot L_i(X) \tag{7}$$

where $(x_i)_{1 \leq i \leq N}$ are collocation points, as roots of the Hermite polynomials available in [18].

Approximation of statistical moments

The mean of the scalar r.v. modelling the mechanical response $Z = g(X)$ is approximated by:

$$\mu_Z \approx \mu_{\tilde{Z}} = \sum_{i=1}^{N} p_X(x_i) \cdot g(x_i) = \sum_{i=1}^{N} \omega_i \cdot g(x_i) \tag{8}$$

where $(\omega_i)_{1 \leq i \leq N}$ are the weights associated with collocation points $(x_i)_{1 \leq i \leq N}$.

The approximation $\sigma_{\tilde{Z}}$ of the standard deviation σ_Z of Z can then be expressed as:

$$\sigma_Z^2 \approx \sigma_{\tilde{Z}}^2 = \sum_{i=1}^{N} (g(x_i))^2 \cdot \omega_i - (\mu_{\tilde{Z}})^2 \tag{9}$$

Approximation of the probability density function

The PDF of the r.v. Z, denoted p_Z, can be approximated by the PDF $p_{\tilde{Z}}$ of the r.v. $\tilde{Z}$, which is an analytical response surface (7). It is thus possible to obtain an estimation of the PDF using Monte Carlo simulations. The curve of $p_{\tilde{Z}}$ is often truncated on an interval $I = [\tilde{z}_{\inf}; \tilde{z}_{\sup}]$, where $\tilde{z}_{\sup/\inf} = \mu_{\tilde{Z}} \pm \alpha \cdot \sigma_{\tilde{Z}}$. In practical terms, α ranges between 3 and 4.

Approximation of an n% confidence interval I_n

The approximated confidence interval for the approximation $\tilde{Z}$ of the r.v. Z, which writes:

$$\tilde{I}_n = [\tilde{z}^n_{\inf}; \tilde{z}^n_{\sup}] \Leftrightarrow \int_{\tilde{z}^n_{\inf}}^{\tilde{z}^n_{\sup}} p_{\tilde{Z}}(z)dz \leq \frac{n}{100} \tag{10}$$

A numerical approximation of the bounds $\tilde{z}^n_{\inf}$ and $\tilde{z}^n_{\sup}$ can ultimately be computed; this approximation delimits the area A on Fig. 1, which displays the evolution of the PDF of the r.v. $\tilde{Z}$.

In practice, only a small number E of input r.v. may be considered, namely 4–5, since the number K of times the mechanical response function f is called increases exponentially with E for a given number N of integration points:

$$K = N^E \tag{11}$$

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