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An efficient SFE method using Lagrange polynomials: Application to nonlinear mechanical problems with uncertain parameters

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Abstract

Designing efficient numerical methods for the solution of stochastic partial differential equations with random inputs or random coefficients is meeting growing interest. So far, the stochastic Galerkin method has been successfully used for various problems with small number of independent random variables. The drawback of this method lies in its difficulty of implementation for nonlinear problems. In this paper we propose a high-order stochastic collocation method to solve nonlinear mechanical systems whose uncertain parameters can be modeled as random variables. Similar to the stochastic Galerkin methods, fast convergence can be achieved when the solution in random space is smooth. However, the numerical implementation of stochastic collocation method is as easy as the Monte-Carlo method since it only requires repetitive runs of an existing deterministic solver. We illustrate the efficiency of this method on two nonlinear mechanical problems in which the random parameters are modeled as correlated lognormal random variables.

Keywords: Stochastic finite elements (SFE); Collocation methods; Hilbertian approximation; Lagrange interpolation

1. Introduction

In structural engineering, the use of the finite element method to simulate complex mechanical problems has reached some degree of maturity. Many sources of errors arising in computer simulations can be controlled and reduced using a posteriori error estimation and mesh adaptivity. All this has increased the accuracy in numerical prediction. While those methods have become increasingly robust and accurate, their reliance on exact data, e.g. material parameters, boundary and initial conditions, geometrical uncertainty, etc., is becoming a bottleneck for the accurate representation of complex problems. In this paper, we strive to design and test an efficient method to compute solutions and their associated sensitivities, assuming that certain properties of the problem are random. There exists two main families of techniques for addressing such problems: sampling methods and non-sampling methods.

One of the most popular sampling methods is probably Monte-Carlo simulations [9,17] or some of its refinements [1,5,6]. In that case, one runs a deterministic code a large number of times and computes the statistics of interest from an ensemble of solutions. The advantage of this approach is its simplicity, however it is difficult to obtain accurate solutions due to its slow rate of convergence $O(N^{-1/2})$ for the mean, where N is the number of samples. Other alternatives exist, such as the quadrature method [2] based on quadrature points and weights, or the SFE method of Baroth et al. [3] (see [20] also, for an extensive comparisons of various methods). However each of those approaches present some drawbacks, for example with the quadrature method, it may be difficult to get an accurate representation of the probability density function.

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Historically, perturbation methods [8] have been one of the first non-sampling methods to be introduced. However, they only work well for mildly nonlinear problems with random inputs and outputs having small variations. Recently, there has been a growing interest for non-sampling methods, where no repetitive deterministic solvers are employed. Some of them are based on the homogeneous chaos theory of Wiener [22]. In its original form, a spectral expansion was employed based on the Hermite orthogonal polynomials in terms of Gaussian random variables. This idea was used fifty years later by Ghanem et al. [11,10] to solve various stochastic problems in mechanics and those methods are usually referred as Stochastic Finite Element Methods (SFEM). Later, other families of polynomials were used to model uncertainty in flow simulations [24], and ideas borrowed from domain decomposition techniques were also applied in [21]. In general, these methods exhibit fast convergence rates with increasing order of the expansions in the random space, provided that the solutions are smooth enough. These methods usually lead to a set of coupled equations that can become very complicated to solve, especially if the underlying differential equations have a nonlinear form.

In this paper, we present and test a stochastic collocation method which combines the strength of non-sampling methods and the ease of implementation of sampling methods. Like some non-sampling methods such as the SFEM, fast convergence can be achieved for smooth solutions in the random space. And similarly to sampling methods, the implementation is straightforward as it only requires solutions of the corresponding problem at each interpolation point. The stochastic collocation method has recently received great attention and was first tested on some test problems in [25]. Then, its efficiency was further demonstrated on more complex problems in electromagnetic [7]. The strong connection between the stochastic collocation method and the pseudo spectral method (see [16], for example) led Babuška et al. [4] to provide rigorous convergence analysis and demonstrate exponential convergence in the probability space for an elliptic partial differential equation with random input. It should be noted that the dimension of the approximating random space grows exponentially with the number of random variables, when tensor product are used to represent it. Therefore, the stochastic collocation method (like the SFEM) will suffer from the so-called curse of dimensionality and this is why the examples of this paper are limited to a maximum of four random variables.

This paper is organized as follow: in Section 2, we start from a general stochastic partial differential equation and we show how it can be discretized using the SFEM (here we give a general description of the technique and we refer it as 'stochastic Galerkin method'). Next, the stochastic collocation method is introduced and the SFE method of [3] is also described. Then, we show how to compute statistics from the representation of the solution and compare with the quadrature method. In Section 3, the method is applied for nonlinear mechanical systems with uncertain parameter and results are compared with the SFE method of Baroth et al. [3]. Finally, in Section 4 we draw conclusions and offer some suggestions for continued research in this direction.

2. Presentation of the method

2.1. Problem setting

In the following, $(\Omega, \mathscr{F}, \mathscr{P})$ denotes a probability space, where Ω is a sample space, $\mathscr{F}$ a σ-algebra of parts of Ω, and $\mathscr{P}$ a probability on $\mathscr{F}$. We assume that the model problem to be solved has the general form

$$\mathscr{L}(\mathbf{Y}, x; u(\mathbf{Y}, x)) = f(\mathbf{Y}, x), \quad x \in D, \tag{1}$$

where D is some bounded domain of $\mathbb{R}^d$ $(d = 1, 2, 3)$ with boundary ∂D, x is the generic point of D , $\mathscr{L}$ is a nonlinear functional, $\mathbf{Y} = (Y^1, \ldots, Y^H)$, $H \geqslant 1$, is a $\mathbb{R}^H$-valued r.v. defined on $(\Omega, \mathscr{F}, \mathscr{P})$, and f and u are two functions from $\mathbb{R}^H \times \mathbb{R}^d$ into $\mathbb{R}$.

In this formulation, the r.v. $\mathbf{Y}$ and the function f are given and the unknown is the $\mathbb{R}$-valued random field $u(\mathbf{Y}, \cdot) = (u(\mathbf{Y}, x), x \in D)$ indexed on D and defined on $(\Omega, \mathscr{F}, \mathscr{P})$ (i.e. $\forall x \in D$, the mapping $\omega \to u(\mathbf{Y}(\omega), x)$ is a real r.v. on $(\Omega, \mathscr{F}, \mathscr{P})$). The problem is then to find the random function $u(\mathbf{Y}, \cdot)$, such that, for $\mathscr{P}$-almost everywhere in Ω, the equation

$$\mathscr{L}(\mathbf{Y}(\omega), x; u(\mathbf{Y}(\omega), x)) = f(\mathbf{Y}(\omega), x), \quad x \in D, \tag{2}$$

is satisfied, subject to boundary conditions on ∂D. We assume that the source terms f and the conditions on the boundary ∂D are regular enough so that the problem is well posed. We also assume that the coordinates $Y^1, \ldots, Y^H$ of the r.v. $\mathbf{Y}$ are mutually independent and that the probability distribution of $\mathbf{Y}$ is absolutely continuous with respect to the Lebesgue's measure on $\mathbb{R}^H$. As a result, the probability density function (pdf) ρ of $\mathbf{Y}$ writes, $\forall y = (y^1, \ldots, y^H) \in \mathbb{R}^H$, $\rho(y) = \Pi_{i=1}^H \rho_i(y^i)$, where ρ_i is the pdf of the random coordinate Y^i. Note that the hypothesis of independence introduced here is not a loss of generality; indeed, it is always possible to construct a change of variable that allows to come down to this situation. Finally, we denote by $\Gamma = \mathbf{Y}(\Omega)$ the image of $\mathbf{Y}$, such that $\Gamma = \Pi_{i=1}^H \Gamma^i$ and $\Gamma^i = Y^i(\Omega)$. With these notations, the general stochastic problem (1) can be rewritten under the following equivalent form: characterize the random field $u(\mathbf{Y}, \cdot) = (u(\mathbf{Y}, x), x \in D)$, where the real function $u = (u(\mathbf{y}, x), (\mathbf{y}, x) \in \Gamma \times D)$ is the solution of the deterministic equation

$$\mathscr{L}(\mathbf{y}, x; u(\mathbf{y}, x)) = f(\mathbf{y}, x), \quad (\mathbf{y}, x) \in \Gamma \times D, \tag{3}$$

with some boundary conditions on ∂D. The above equation can be solved with standard discretization techniques. As is usually done in deterministic problems, we seek the numerical solution $u_h(\mathbf{y}, x)$ in a finite-dimensional subspace $V_h \subset L^2_\rho(\Gamma)$ that satisfies the weak equation

$$\int_\Gamma \mathscr{L}(\mathbf{y},x;u_h(\mathbf{y},x))\phi_h(\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_\Gamma f(\mathbf{y},x)\phi_h(\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}, \quad \forall\phi_h \in V_h. \tag{4}$$

In this paper, we decide to choose V_h as being the subspace of polynomials of degree P at most. Depending on the construction of the subspace V_h, Eq. (4) will lead to different numerical schemes which are discussed below. In the following, for the sake of simplicity, the r.v. $\mathbf{Y}$ will be assumed to be scalar (i.e. $H=1$), the extension to higher dimensional spaces by tensor product being trivial. However, for the examples of Section 3 we shall take $H=2$ and $H=4$.

2.1.1. Stochastic Galerkin methods

In that case, we choose to span V_h by a set of orthonormal polynomials with respect to the weight function $\rho(y)$, i.e. we have $V_h = \mathrm{span}\{\phi_m(y)\}_{0\leqslant m\leqslant P}$ where $\phi_m(y)$ are degree P polynomials satisfying

$$\int_\Gamma \phi_m(y)\phi_n(y)\rho(y)\,\mathrm{d}y = \delta_{mn}; \quad m,n=0,\ldots,P. \tag{5}$$

For Gaussian distributions, those are known as Hermite polynomials whereas for uniform distributions they are Legendre polynomials (see [23] for a complete description of different families of polynomials). However, it should be noted that one can choose any family of polynomials as long as $\{\phi_m(y)\}_{0\leqslant m\leqslant P}$ form a complete basis. Nevertheless, if the stochastic solution stemming from Eq. (4) has a probability density close to $\rho(y)$, we can expect faster convergence of the numerical solution by taking orthonormal polynomials with respect to the weight function $\rho(y)$. We seek the numerical solution of Eq. (4) under the form

$$u_h(y,x) = \sum_{i=0}^{P} u_h^i(x)\phi_i(y). \tag{6}$$

This is known as a modal representation of $u_h(y,x)$, where $u_h^i(x)$ are the modes that need to be computed. This is done by inserting the expression (6) into (4) to obtain

$$\int_\Gamma \mathscr{L}\left(y,x;\sum_{i=0}^{P} u_h^i(x)\phi_i(y)\right)\phi_j(y)\rho(y)\,\mathrm{d}y = \int_\Gamma f(y,x)\phi_j(y)\rho(y)\,\mathrm{d}y; \quad j=0,\ldots,P. \tag{7}$$

In the above equation, the integrals can be computed with numerical quadrature rules, however we see that if $\mathscr{L}$ is a nonlinear operator, all modes $u_h^i(x)$ are coupled and this leads to a very complicated system that need to be solved. This approach was used in many stochastic problems in mechanics by Ghanem [11] and the expression (6) is usually referred as polynomial Chaos expansion. Many other methods can be designed around the same ideas, by choosing different basis functions in (6), for example. In the next subsection, we will show that other choices of basis can lead to uncoupled systems, even if $\mathscr{L}$ is a nonlinear operator.

2.1.2. Stochastic collocation methods

We now choose to span V_h by a set of Lagrange interpolating polynomials of degree P, i.e. $V_h = \mathrm{span}\{L_m(y)\}_{0\leqslant m\leqslant P}$, where the Lagrange polynomials are based on the interpolating points $\{y^m\}_{0\leqslant m\leqslant P}$. We will see that a particularly convenient choice of points that will considerably simplify equations consists in choosing the quadrature points of the formula

$$\int_\Gamma f(y)\rho(y)\,\mathrm{d}y = \sum_{i=0}^{P} w^i f(y^i). \tag{8}$$

The points y^i and weights w^i are built such that the formula (8) is exact for all polynomials of degrees up to $(2P+1)$. They can be obtained by solving a nonlinear system with Newton's method (see [15] for example). We now seek the numerical solution of (4) under the form

$$u_h(y,x) = \sum_{i=0}^{P} u_h^i(x)L_i(y), \tag{9}$$

and this is known as a nodal representation of $u_h(y,x)$, where $u_h^i(x) = u_h(y^i,x)$ is the solution of (4) at node y^i (this is also referred as 'surface response' in the mechanical field, see [20]). To compute $u_h^i(x)$, we insert the expression (9) into (4) and use the quadrature rule (8) to evaluate the integrals:

$$\sum_{k=0}^{P} w^k \mathscr{L}\left(y^k,x;\sum_{i=0}^{P} u_h^i(x)L_i(y^k)\right)L_j(y^k) = \sum_{k=0}^{P} w^k f(y^k,x)L_j(y^k); \quad j=0,\ldots,P. \tag{10}$$

An important property of Lagrange polynomials that will considerably simplify the expression above is that $L_j(y^k)=\delta_{jk}$, so we get

$$\mathscr{L}(y^j,x;u_h^j(x)) = f(y^j,x); \quad j=0,\ldots,P. \tag{11}$$

Therefore, the coefficients $\{u_h^j(x)\}_{0\leqslant j\leqslant P}$ of (9) are simply computed by solving $(P+1)$ decoupled Eq. (11) similar to the original one (1), with $(P+1)$ realizations corresponding to $\{y^j\}_{0\leqslant j\leqslant P}$, just like a Monte-Carlo simulation. A comparison of the Galerkin method and the collocation method [7] shows that for smooth problems, both should exhibit fast convergence rates. It is only for problems with very low regularity that the stochastic Galerkin method might be more attractive.

2.1.3. The SFE method of Baroth et al. [3]

In the next section, we shall compare numerical results using the stochastic collocation method with those obtained with the method of Baroth et al. presented in [3]. We briefly recall the elements of this technique, which borrows ideas from the stochastic collocation method and the stochastic Galerkin method. First, a piecewise

polynomial approximation of $u(y,x)$ is constructed by collocation to represent the stochastic solution $\widetilde{u}_h(y,x)$:

$$\widetilde{u}_h(y,x) = \sum_{i=0}^{M} \widetilde{u}_h^i(x) S_i(y), \tag{12}$$

where $S_i(y)$ are cubic B-splines. This approximation is then projected on to the orthonormal polynomial basis $\{\phi_i(y)\}_{0\leqslant i\leqslant P}$ defined in Section 2.1.1

$$u_h(y,x) = \sum_{i=0}^{P} u_h^i(x)\phi_i(y), \tag{13}$$

where, the coefficients $u_h^i(x)$ are given by

$$u_h^i(x) = \int_\Gamma \widetilde{u}_h(y,x)\phi_i(y)\rho(y)\,\mathrm{d}y. \tag{14}$$

This last projection was done in order to facilitate the computation of statistics and reconstruct the probability density function of the stochastic solution by performing a Monte-Carlo simulation from (13). Note that the final solution (13) is given under the same form as Eq. (6), without having to solve a large problem which coupled coefficients $u_h^i(x)$ as would have happened with the stochastic Galerkin method. On the other hand, this double projection (first with cubic B-splines and then on an orthogonal polynomial basis) might be harmful for the accuracy of the solution, as we shall see on numerical examples.

2.2. Computation of statistics and probability density function

Let $u_h(Y,\cdot) = (u_h(Y,x), x \in D)$ be the approximate stochastic solution stemming from (6) or (9). In practice, one is often interested in the first two moments i.e. the mean value and the variance. Using the orthonormality of the polynomials $\{\phi_i(y)\}_{0\leqslant i\leqslant P}$, it is easy to show [7] from (6) or (13) that the mean value of the r.v. $u_h(Y,x)$ is simply the first mode in the expansion, i.e. $\langle u_h(Y,x)\rangle = u_h^0(x)$ and the variance is

$$\mathrm{var}(u_h(Y,x)) = \sum_{i=1}^{P} (u_h^i(x))^2. \tag{15}$$

When the deterministic solution is given under the form (9), using the property of Lagrange polynomials $L_j(y^k) = \delta_{jk}$ and the quadrature formula (8), we can show [7] that

$$\langle u_h(Y,x)\rangle = \sum_{i=0}^{P} u_h^i(x) \int_\Gamma L_i(y)\rho(y)\,\mathrm{d}y = \sum_{i=0}^{P} w^i u_h^i(x) \tag{16}$$

and

$$\begin{aligned}\langle u_h(Y,x)^2\rangle &= \sum_{i,j=0}^{P} u_h^i(x)u_h^j(x) \int_\Gamma L_i(y)L_j(y)\rho(y)\,\mathrm{d}y \\ &= \sum_{i=0}^{P} w^i (u_h^i(x))^2.\end{aligned} \tag{17}$$

For the applications of Section 3, we are also interested in the skewness and the kurtosis. Therefore, we need similar formula for higher order moments. There is no quadrature error for the evaluation of the first two moments (16) and (17) since the quadrature formula (8) is exact for polynomials up to order $(2P+1)$ and we integrate polynomials of order P and $2P$, respectively. For higher order moments, we need more accurate quadrature formulas to ensure exact integration. In that case, the moments of order k $(k>2)$, are given by

$$\langle u_h(Y,x)^k\rangle = \sum_{i_1\ldots i_k=0}^{P} u_h^{i_1}(x)\ldots u_h^{i_k}(x) \sum_{l=0}^{M} w^l L_{i_1}(y^l)\ldots L_{i_k}(y^l), \tag{18}$$

where M is the integer part of $(kP/2)$, so that the quadrature points and weights $(y^l, w^l)_{0\leqslant l\leqslant M}$ ensure exact integration. It should be noted that computing any moment of order k with the quadrature formula (8) would give

$$\langle u_h(Y,x)^k\rangle = \sum_{i=0}^{P} w^i (u_h^i(x))^k. \tag{19}$$

Although this expression has the advantage of simplicity, we are integrating a polynomial of degree kP with a quadrature formula exact up to polynomial degrees $(2P+1)$ and therefore, as k increases, the computation of $\langle u_h(Y,x)^k\rangle$ will become less and less accurate.

One particularly interesting feature of the methods presented in that section is that accurate probability density functions can be obtained at low cost by doing Monte-Carlo simulations from (6) or (9).

2.3. Comparison with the quadrature method

In this subsection, we show that there are strong connections between the stochastic collocation method and the quadrature method presented in [2]. Let us first briefly recall the principle of this method. The idea is to use quadrature formula to compute moments of different orders of the response quantities. Assuming that the joint probability density of a random variable Y is ρ, and that we are interested in the response $u_h(Y,x)$, Baldeweck [2] (see also [14]) was approximating the moments of $u_h(Y,x)$ using the same formula (19), i.e.

$$\langle u_h(Y,x)^k\rangle = \sum_{i=0}^{P} w^i (u_h(x,y^i))^k, \tag{20}$$

where $(y^i, w^i)_{0\leqslant i\leqslant P}$ are the integration points and weights associated with ρ. The first four moments were computed and a probability density function of the response was constructed from those moments using Johnson's method, for example (see [12,2] for details). We see that the computation of the first two moments gives strictly identical results for the quadrature method and the stochastic collocation method. For higher order moments, Eq. (18) will be more accurate since it integrates exactly the moments of any or-

der of the response given under the form (9). Another drawback of the quadrature method lies in the difficulty to construct accurate probability density functions of the response from its statistical moments, when P is small.

3. Applications

In this section, we illustrate the efficiency of the proposed method for two nonlinear mechanical problems. For the first example (sphere under internal pressure), there is no variation of the solution in the physical space, i.e. there is no x variable in Eq. (1). The second example (Hertz contact problem) is a nonlinear problem for which a discretization is performed both in the physical space and in the probability space. The uncertain parameters are modeled as lognormal random variables which are correlated. The quadrature points and weights for the lognormal law can be computed from those of the standard normal law (y^i, w^i) by a simple change of variable, and it can be easily shown that they are (e^{y^i}, w^i). From an implementation point of view, two equivalent approaches can be adopted. We can keep the lognormal r.v. in the equations and use the quadrature points and weights (e^{y^i}, w^i) associated with the lognormal law. Alternatively, we can transform the lognormal r.v. into standard Gaussian ones in the equations and use the quadrature points and weights (y^i, w^i) associated with the normal law. In this paper, we have adopted this last procedure.

3.1. Sphere under internal pressure

3.1.1. Presentation of the problem

We consider a hollow sphere under internal pressure (see Fig. 1) and we assume that the constituent material is elastic perfectly plastic. The geometrical and mechanical parameters of the model are the internal and external radius a and b, the Young modulus E, the Poisson ratio ν, the yield stress f_e and the internal pressure p. For this first problem, we assume that the two random parameters are the Young modulus E and the Poisson ratio ν, afterwards denoted by z_1 and z_2. They are modeled as a bidimensional lognormal r.v. $\mathbf{Z} = (Z_1, Z_2)$ with mean $(\mu_{Z_1}, \mu_{Z_2}) = (2 \times 10^{11}\ \text{Pa}, 0.3)$ and standard deviation $(\sigma_{Z_1}, \sigma_{Z_2}) = (\mu_{Z_1} a_{Z_1}, \mu_{Z_2} a_{Z_2})$. In order to illustrate the robustness of the method with regard to the scattering of the input variables, we choose the coefficients of variation of $\mathbf{Z}$ equal to $(a_{Z_1}, a_{Z_2}) = (0.3, 0.1)$. The coefficient of correlation $\rho_{Z_1 Z_2}$ between Z_1 and Z_2 is chosen equal to 0.8 and the elements of the covariance matrix are given by $C_{Z_i Z_j} = \sigma_{Z_i} \sigma_{Z_j} \rho_{Z_i Z_j}$, with $\rho_{Z_1 Z_1} = \rho_{Z_2 Z_2} = 1$. The other parameters of this academic example are assumed to be deterministic and equal to $a = 10^{-3}$ m, $b = 2 \times 10^{-3}$ m, $f_e = 3 \times 10^8$ Pa and $p = 3.589 \times 10^8$ Pa. We are interested here in the radial plastic displacement u at any point of the internal outline. The deterministic solution of this problem is known [13]. In the random case, the solution is a scalar r.v. U given by

$$U = F(\mathbf{Z}), \tag{21}$$

where F is a mapping from $\mathbb{R}_+^* \times \mathbb{R}_+$ into $\mathbb{R}$, such that, $\forall \mathbf{z} = (z_1, z_2) \in \mathbb{R}_+^* \times \mathbb{R}_+$

$$u = F(\mathbf{z}) = \frac{a f_e [(1 - z_2)\alpha^3 + 2(2z_2 - 1)(\ln(\alpha) + (1/3)(1 - \beta^3))]}{z_1}, \tag{22}$$

with

$$\alpha = \frac{c}{a} \quad \text{and} \quad \beta = \frac{c}{b}. \tag{23}$$

The c constant above is obtained by solving the equation

$$p = 2 f_e \left[\ln \alpha + \frac{1}{3}(1 - \beta^3) \right]. \tag{24}$$

Outcomes (z_1, z_2) of the bidimensional lognormal random vector $\mathbf{Z}$ can be generated from outcomes (y_1, y_2) of a bidimensional standard Gaussian random vector $\mathbf{Y} = (Y_1, Y_2)$ as follows:

$$\begin{cases} z_1 = \frac{\mu_{Y_1}}{\sqrt{1 + a_{Y_1}^2}} \exp(l_{11} y_1), \\ z_2 = \frac{\mu_{Y_2}}{\sqrt{1 + a_{Y_2}^2}} \exp(l_{21} y_1 + l_{22} y_2), \end{cases} \tag{25}$$

where the coefficients l_{11}, l_{21} and l_{22} are given in the Appendix A of [3]. The expression (22) is now a function of (y_1, y_2) and we can write $F(z_1, z_2) = u(y_1, y_2)$. For this problem, the algorithm of the stochastic collocation method described in the previous section is summarized below.

- Generate $(P + 1)$ quadrature points and weights (y^i, w^i) associated with the standard Gaussian law.
- Compute the $(P + 1)^2$ outcomes $u^{ij} = u(y_1^i, y_2^j)$ for $0 \leqslant i, j \leqslant P$.
- The numerical solution (or surface response) writes

$$u_h(y_1, y_2) = \sum_{i=0}^{P} \sum_{j=0}^{P} u^{ij} L_i(y_1) L_j(y_2), \tag{26}$$

where $L_i(y)$ are Lagrange polynomials of degree P based on points $\{y^i\}_{0 \leqslant i \leqslant P}$.

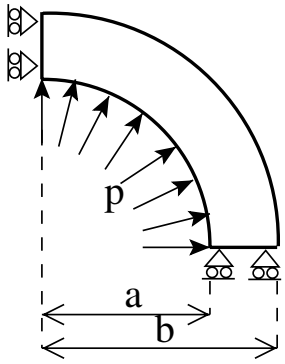

Fig. 1. Sphere under internal pressure.

- The first two moments of the approximate stochastic solution $u_h(Y_1, Y_2)$ are $\langle u_h(Y_1, Y_2)\rangle = \sum_{i=0}^{P}\sum_{j=0}^{P} w^i w^j u^{ij}$ and $\langle (u_h(Y_1, Y_2))^2\rangle = \sum_{i=0}^{P}\sum_{j=0}^{P} w^i w^j (u^{ij})^2$. Higher order integration formulas have to be used for moments of order greater than two (see Eq. (18)).
- The probability density function can be estimated from (26) by using a Monte-Carlo approach.

3.1.2. Convergence of statistical moments

We compare the moments of the solution (21) with three different methods: the SFE method presented in [3], the quadrature method [2] and the stochastic collocation method introduced in the previous section. We are interested in the convergence of the mean, the variance, the skewness and the kurtosis of the response U. We plot the logarithm of the error for those moments, as a function of the number of calls N of the function F defined by Eq. (22). For the SFE method N is the number interpolation points for a two-dimensional problem, i.e. $N = N_I^2$, where N_I is the number of B-spline interpolation points in each dimension (see [3] for details). Similarly, for the quadrature method and the stochastic collocation method with Lagrange polynomials, we have $N = (P+1)^2$ where $(P+1)$ is the number of quadrature or collocation points in each direction. The reference solution used to compute the relative errors was obtained with the stochastic collocation method by taking $N = 196$.

Figs. 2 and 3 represent the error on the mean and the variance as N increases. We can see that exponential convergence is achieved with the stochastic collocation method, up to machine's precision. On the other hand, the convergence of the SFE method [3] is much slower. Note that for those two moments, the quadrature method gives results identical to the stochastic collocation method, as explained in the previous section.

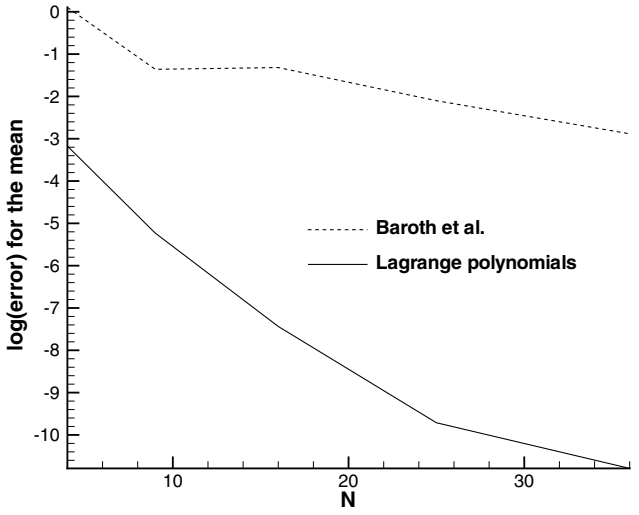

Fig. 2. Representation of the logarithm of the error as a function of N for the mean. Results for the SFE method of Baroth et al. [3] and the stochastic collocation method are shown for the sphere problem.

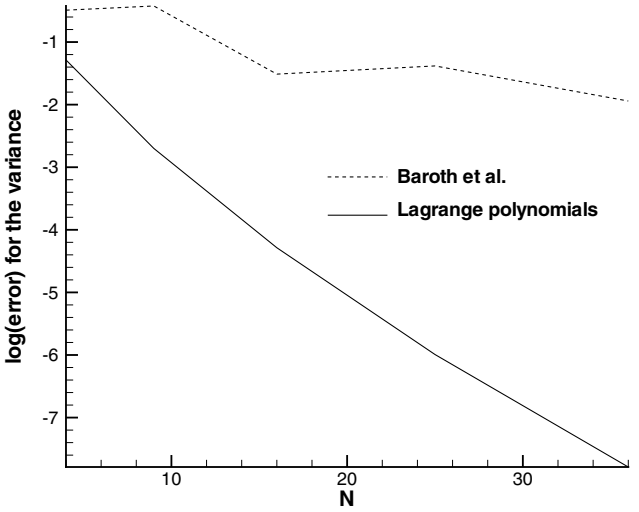

Fig. 3. Representation of the logarithm of the error as a function of N for the variance. Results for the SFE method of Baroth et al. [3] and the stochastic collocation method are shown for the sphere problem.

The convergence of the solution for the skewness and the kurtosis is shown in Figs. 4 and 5. We see that both the quadrature method and the stochastic collocation method exhibit fast convergence. However, as we integrate higher order moments, the stochastic collocation becomes more accurate.

3.1.3. Comparison of the probability density function

We now compare the pdf given by four different methods: the SFE method [3], the quadrature method, the collocation method using Lagrange polynomials and finally the Monte-Carlo method. For the Monte-Carlo method, we use 10^5 samples to construct the pdf. The quadrature method only gives moments of any order of the response U.

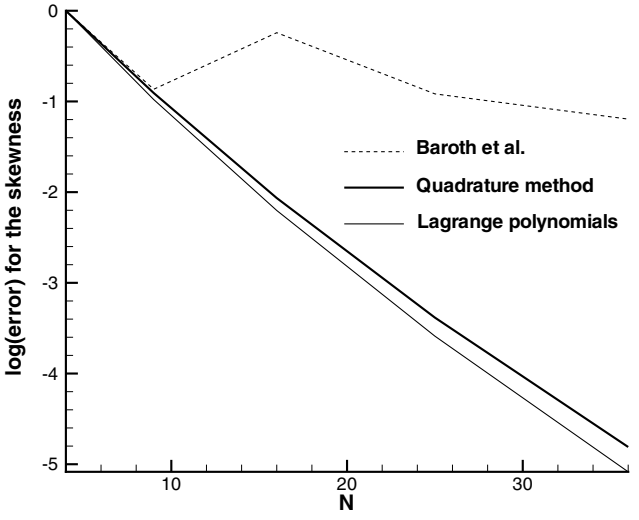

Fig. 4. Representation of the logarithm of the error as a function of N for the skewness. Results for the SFE method of Baroth et al. [3], the quadrature method and the stochastic collocation method are shown for the sphere problem.

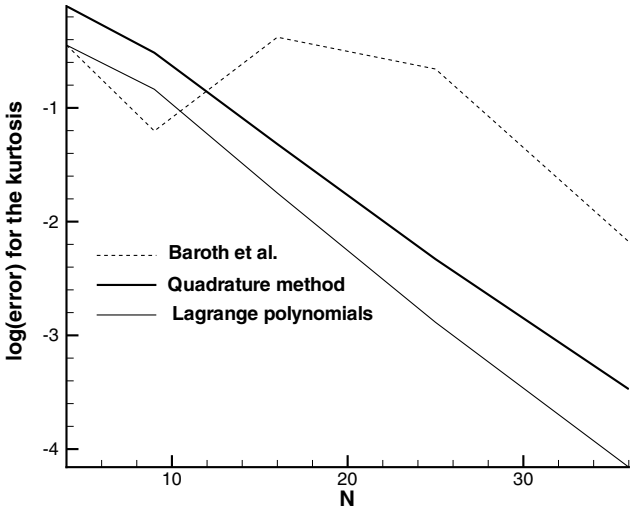

Fig. 5. Representation of the logarithm of the error as a function of N for the kurtosis. Results for the SFE method of Baroth et al. [3], the quadrature method and the stochastic collocation method are shown for the sphere problem.

Therefore, the pdf has to be constructed from this information only, and this is done using Johnson's method based on the first four moments (see [2] for details). For the SFE method, the pdf is constructed by doing a Monte-Carlo simulation from the solution written under the form (6). This procedure is efficient since evaluating samples solutions from (6) is an inexpensive computation. For the stochastic collocation method, the same procedure is applied, but using the expression (9). In both cases, the pdf is constructed using 10^7 samples.

Fig. 6 shows the pdf obtained with a small value of N. In that case, the stochastic collocation method is the only one which gives a pdf close to the exact one. This result were obtained with only four calls ($N = 4$) to the response function F given by (22).

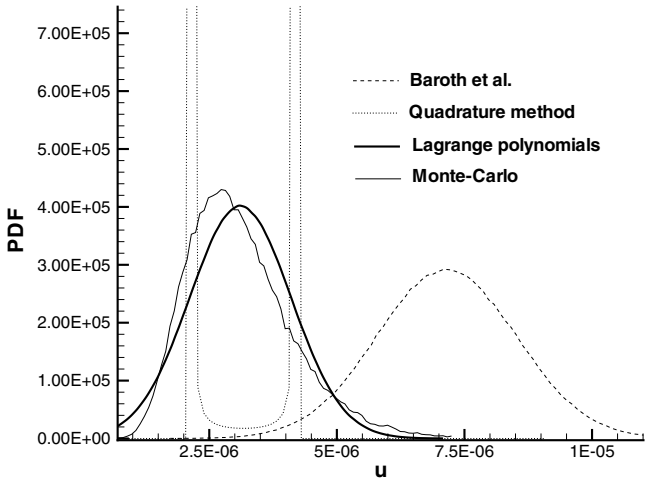

Fig. 6. Representation of the pdf for the sphere problem using four different methods for $N = 4$.

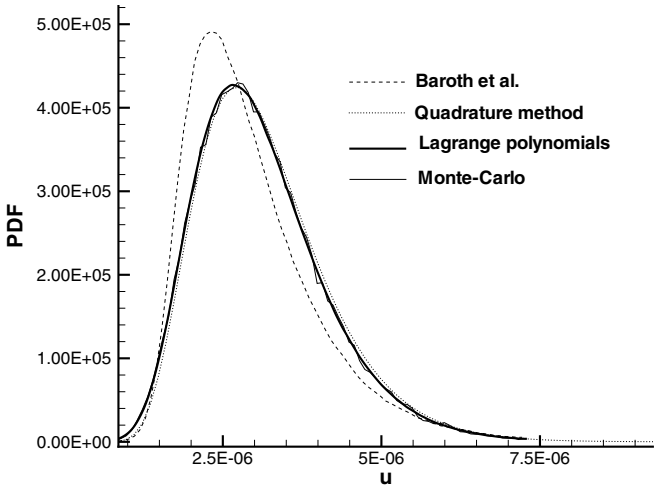

Fig. 7. Representation of the pdf for the sphere problem using four different methods for $N = 16$.

When N is increased up to $N = 16$ in Fig. 7, we see that with the exception of the SFE method [3], all methods give good results. Finally, for $N = 36$, all methods have converged and give similar results (see Fig. 8).

3.2. Elasto-plastic cylinder

3.2.1. Presentation of the problem

As a second example, we consider a Hertz contact problem between an infinitely long cylinder and a fixed rigid horizontal plane. The cylinder is compressed by a vertical uniform load along its axis and we are interested in the vertical displacements (u_i) of the section S ($\mathbf{1} = (0, 2R)$, $\mathbf{2} = (0, \frac{3}{2}R)$, $\mathbf{3} = (0, R)$, $\mathbf{4} = (0, \frac{2}{3}R)$, $\mathbf{5} = (0, \frac{1}{3}R)$) (see Fig. 9). These displacements are gathered in a vector $\mathbf{u}$. The behaviour of the material is supposed to be isotropic bilinear elasto-plastic. Plane strain assumption is made, so the analysis can be reduced to a two-dimensional

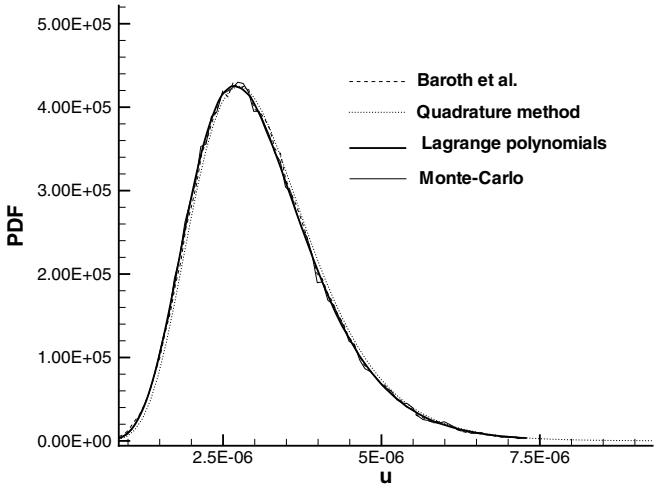

Fig. 8. Representation of the pdf for the sphere problem using four different methods for $N = 36$.

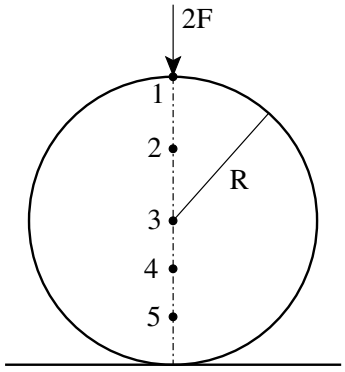

Fig. 9. Geometry of the Hertz contact problem.

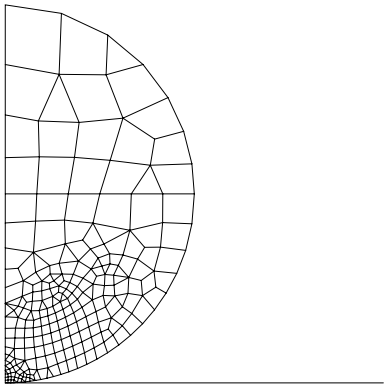

Fig. 10. Finite element mesh for the Hertz contact problem.

problem. Thanks to the problem symmetry, only half section of the cylinder is discretized, using linear plain strain Finite Elements (see Fig. 10). Contact is taken into account using a mesh of contact FE, linking the basis of the cylinder and the portion of the plane that may be in contact.

This mechanical problem is nonlinear and the FE resolution is incremental and iterative. Parameters of the deterministic model, namely the Poisson's ratio ν, the half-loading intensity F_l, the radius R and the yield limit stress f_e, are equal to $\nu = 0.3$; $F_\mathrm{l} = 5 \times 10^3$ N; $R = 5 \times 10^{-2}$ m and $f_e = 3 \times 10^7$ Pa. The uncertain parameters of this model is the Young modulus E and the plastic modulus E_p, denoted afterwards z_1 and z_2 respectively. As before, the couple (z_1, z_2) is modeled as a two-dimensional lognormal r.v. $\mathbf{Z} = (Z_1, Z_2)$, with characteristics $(\mu_{Z_1}, \mu_{Z_2}) = (3 \times 10^{10}$ Pa, 9×10^9 Pa), $a_{Z_1} = a_{Z_2} = 0.1$ and $\rho_{Z_1 Z_2} = 0.9$. As a result, the vector displacements $\mathbf{u}$ is a vector r.v. that we shall denote by $\mathbf{U} = (U_1, \ldots, U_5)$. In the sequel, we will show results for the random variable U_5, obtained after the final load F_l.

3.2.2. Convergence of statistical moments

Figs. 11–14 show the convergence of the mean, the variance, the skewness and the kurtosis, respectively, as a function of the number of calls N to the finite element model, which is directly related to the CPU time. Here again, the reference solution used to compute the errors was obtained with the stochastic collocation method by taking $N = 196$.

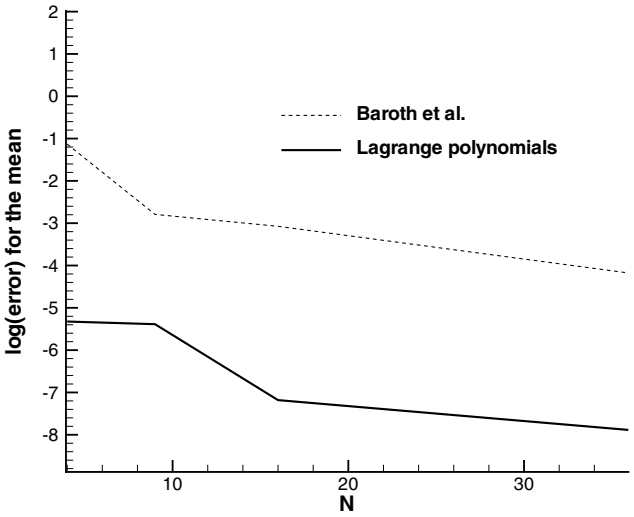

Fig. 11. Representation of the logarithm of the error as a function of N for the mean. Results for the SFE method of Baroth et al. [3] and the stochastic collocation method are shown for the cylinder problem.

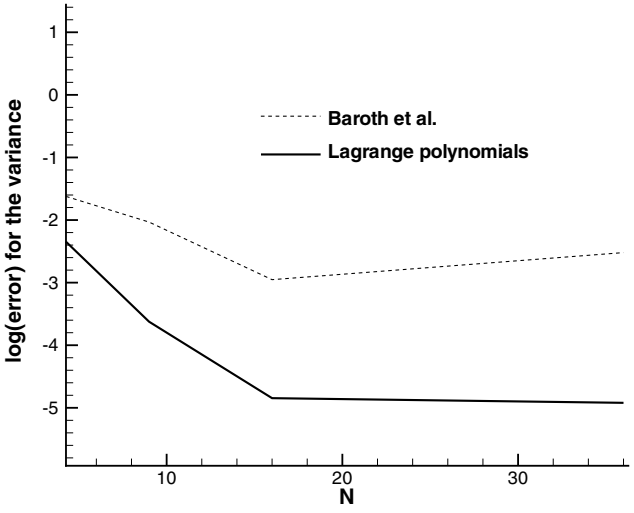

Fig. 12. Representation of the logarithm of the error as a function of N for the variance. Results for the SFE method of Baroth et al. [3] and the stochastic collocation method are shown for the cylinder problem.

For a fixed value of N, both the quadrature method and the stochastic collocation method give better results than the method of Baroth et al. However, we observe that the convergence is not as fast as in the previous example. A possible reason is that the total error comes not only from the discretization in the probability space but also from the discretization in the physical space. From $N = 16$, no significant improvement of the accuracy can be observe by increasing the value of N because the error coming from the finite element discretization in the physical space becomes predominant.

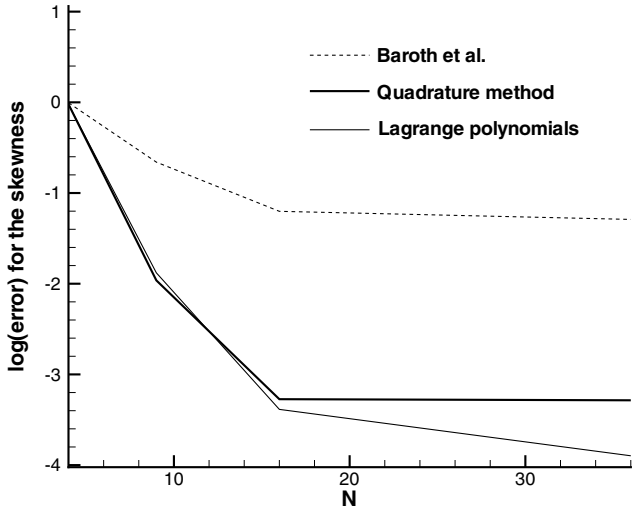

Fig. 13. Representation of the logarithm of the error as a function of N for the skewness. Results for the SFE method of Baroth et al. [3], the quadrature method and the stochastic collocation method are shown for the cylinder problem.

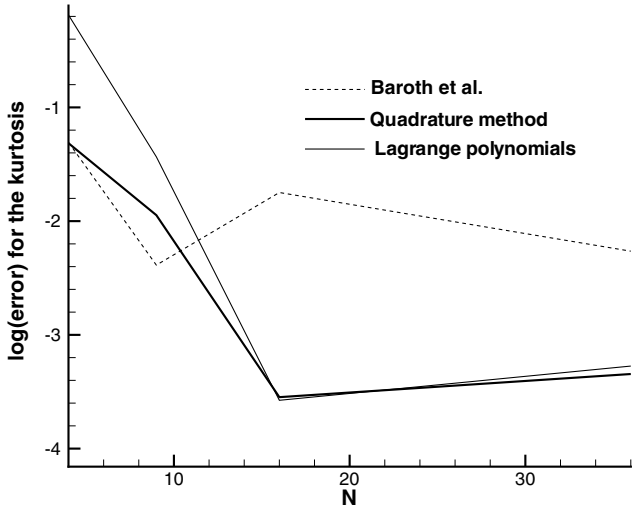

Fig. 14. Representation of the logarithm of the error as a function of N for the kurtosis. Results for the SFE method of Baroth et al. [3], the quadrature method and the stochastic collocation method are shown for the cylinder problem.

3.2.3. Comparison of the probability density function

Similarly to the sphere problem, Figs. 15–17 show the pdf given by the SFE method [3], the quadrature method and the stochastic collocation for different levels of discretization in the probability space. The pdf given by the Monte-Carlo method using 10^3 samples is also shown on those three figures. We observe that even at a low level of discretization ($N = 4$), the stochastic collocation method gives very good results. For $N = 36$, all methods have converged with the exception to the method of Baroth et al.

Those results suggest that it is possible to use the stochastic collocation method efficiently, even for problems with more than two random variables. This is what we

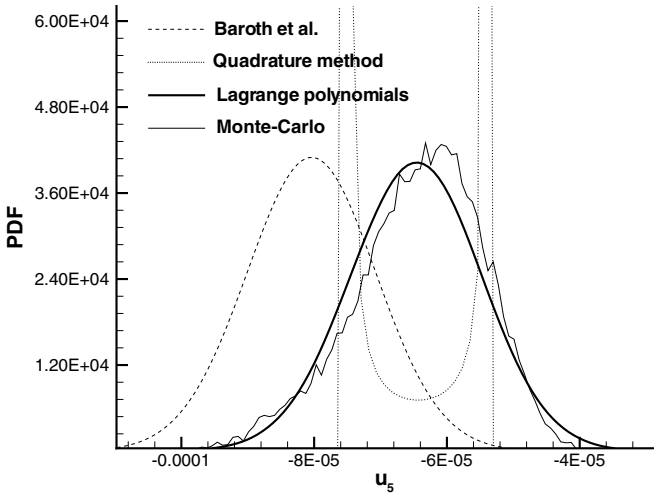

Fig. 15. Representation of the pdf for the cylinder problem using four different methods for $N = 4$.

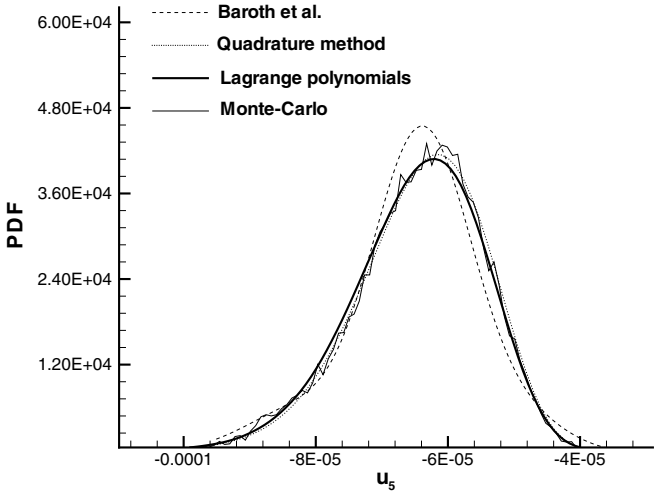

Fig. 16. Representation of the pdf for the cylinder problem using four different methods for $N = 16$.

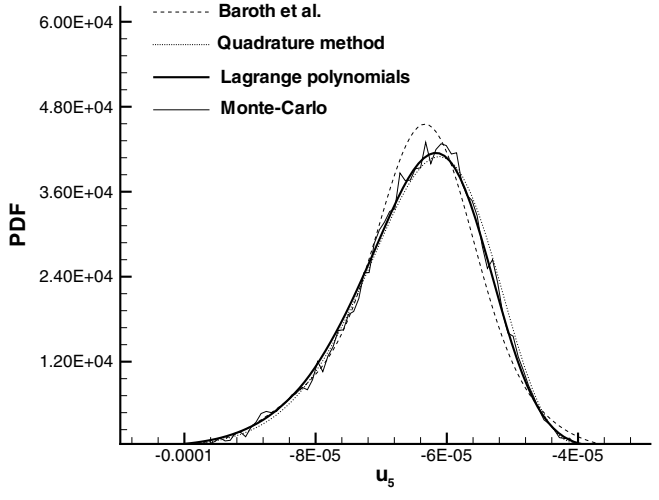

Fig. 17. Representation of the pdf for the cylinder problem using four different methods for $N = 36$.

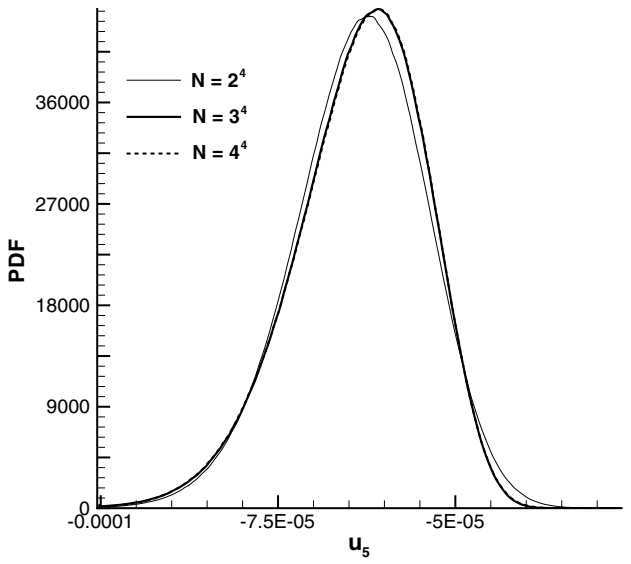

Fig. 18. Representation of the pdf for the Hertz contact problem with four random variables. The number of calls to the mechanical function is $N = 2^4$, 3^4 and 4^4.

do in the next subsection, where the Hertz problem is solved in the case of four random variables.

3.2.4. Numerical test with four random variables

We now solve the Hertz contact problem with four random variables (Z_1, Z_2, Z_3, Z_4) where Z_1 and Z_2 are the same as before; Z_3 and Z_4 are two independent r.v. used to model the uncertainty of the radius R of the cylinder and the half-loading intensity F_l, respectively. All the deterministic parameters are the same as before and for the random variables, we take $(\mu_{Z_1}, \mu_{Z_2}, \mu_{Z_3}, \mu_{Z_4}) = (3 \times 10^{10}\ \text{Pa}, 9 \times 10^9\ \text{Pa}, 0.05\ \text{m}, 3 \times 10^5\ \text{N})$, $a_{Z_1} = a_{Z_2} = a_{Z_3} = a_{Z_4} = 0.1$ and $\rho_{Z_1 Z_2} = 0.9$.

Fig. 18 shows the pdf of the vertical displacements u_5 of the cylinder for three levels of discretization in the probability space. In each of the four directions, 2, 3 and then 4 collocations points are used, leading to a number of calls N to the mechanical function equal to 2^4, 3^4 and 4^4, respectively. We see that even at a low level of discretization $(N = 2^4)$, the pdf is already very close to the converged solution. No distinction can be made between the solution obtained with $N = 3^4$ and $N = 4^4$, suggesting that very accurate solutions can be obtained with only three points of collocation in each direction.

4. Conclusion

In this paper, we have proposed an efficient SFE method based on Lagrange interpolating polynomials. The method was tested and compared with other techniques for the resolution of two nonlinear stochastic mechanical problems. Its formulation is very attractive since it is similar to that of a Monte-Carlo method, where a deterministic equation is solved repetitively at each discrete point in the random space. We have shown that fast convergence can be achieved for the moments and accurate pdf could be computed. For a given number of collocations points, the method compares favourably with the others tested methods. Throughout this paper, dimensions of the random parameter up to four were tested. For more general problems of dimension H, the cost of this method would be proportional to H^{P+1}. Even if good accuracy can be obtained with low values of P (typically, $P = 3$ was shown to be sufficient), this method (like the stochastic Galerkin method) will suffer from the curse of dimensionality. Therefore, we should not realistically expect it to be efficient for random parameters of dimension larger than five or six. For larger dimensions of the random parameter, high-dimensional integration formulas [18,19] might be a good alternative and this is ongoing research.

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