Chapter IV

Solving linear and nonlinear partial differential equations by the method of characteristics

Chapter III has brought to light the notion of characteristic curves and their significance in the process of classification of partial differential equations.

Emphasis will be laid here on the role of characteristics to guide the propagation of information within hyperbolic equations. As a tool to solve PDEs, the method of characteristics requires, and provides, an understanding of the structure and key aspects of the equations addressed. It is particularly useful to inspect the effects of initial conditions, and/or boundary conditions.

While the method of characteristics may be used as an alternative to methods based on transform techniques to solve linear PDEs, it can also address PDEs which we call quasi-linear (but that one usually coins as nonlinear). In that context, it provides a unique tool to handle special nonlinear features, that arise along shock curves or expansion zones.

As a model problem, the method of characteristics is first applied to solve the wave equation due to disturbances over infinite domains so as to avoid reflections. The situation is more complex in semi-infinite or finite bodies where waves get reflected at the boundaries. The issue is examined in Exercise IV.2.

Basic features of scalar conservation laws are next addressed with emphasis on under- and over-determined characteristic network, associated with expansion zone and shock curves.

Finally some guidelines to solve PDEs via the method of characteristics are provided. Unlike transform methods, the method is not automatic, is a bit tricky and requires some experience. [1]

IV.1 Waves generated by initial disturbances

IV.1.1 An initial value problem in an infinite body

For an infinite elastic bar, aligned with the axis x,

$$-\infty \quad \cdots \underline{\overline{\qquad\qquad\qquad}} \cdots \quad +\infty \tag{IV.1.1}$$

[1]Posted, December 12, 2008; Updated, April 24, 2009

the field equation describing the homogeneous wave equation,

$$\text{(FE)} \quad \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0\,, \quad -\infty < x < \infty, \quad t > 0\,, \tag{IV.1.2}$$

with

$$\begin{aligned} &u = u(x,t) \quad \text{unknown axial displacement,} \\ &(x,t) \quad \text{variables}\,, \end{aligned} \tag{IV.1.3}$$

is complemented by Cauchy initial conditions,

$$\begin{aligned} &(\text{CI})_1 \quad u(x, t=0) = f(x), \quad -\infty < x < \infty \\ &(\text{CI})_2 \quad \frac{\partial u}{\partial t}(x, t=0) = g(x), \quad -\infty < x < \infty\,, \end{aligned} \tag{IV.1.4}$$

and conditions at infinity,

$$\begin{aligned} &(\text{CL})_1 \quad u(x \to \pm\infty,\, t) = 0 \\ &(\text{CL})_2 \quad \frac{\partial u}{\partial t}(x \to \pm\infty,\, t) = 0\,. \end{aligned} \tag{IV.1.5}$$

Written in terms of the characteristic coordinates,

$$\xi = x - c\,t, \quad \eta = x + c\,t\,, \tag{IV.1.6}$$

the *canonical form* (IV.1.2) transforms into the *canonical form*,

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0\,, \quad -\infty < x < \infty, \quad t > 0\,, \tag{IV.1.7}$$

whose general solution,

$$\begin{aligned} u(\xi, \eta) &= \phi(\xi) + \psi(\eta) \\ &= \phi(x - c\,t) + \psi(x + c\,t)\,, \end{aligned} \tag{IV.1.8}$$

expresses in terms of two arbitrary functions to be defined.

One could not stress enough the interpretation of these results.

Along a characteristic $\xi = x - c\,t$ constant, the solution $\phi(\xi)$ keeps constant: for an observer moving to the right at speed c, the initial profile $\phi(\xi)$ keeps identical. A similar interpretation holds for the part of the solution contained in $\psi(\eta)$ which propagates to the left.

We now will consider the effects of the initial conditions, so as to obtain the two unknown functions ϕ and ψ.

IV.1.2 D'Alembert solution

The initial conditions,

$$\begin{aligned} &(\text{CI})_1 \quad u(x, t=0) = f(x) = \phi(x) + \psi(x), \quad -\infty < x < \infty \\ &(\text{CI})_2 \quad \frac{\partial u}{\partial t}(x, t=0) = g(x) = -c\,\frac{d\phi}{dx} + c\,\frac{d\psi}{dx}, \quad -\infty < x < \infty\,, \end{aligned} \tag{IV.1.9}$$

imply

$$-\phi(x) + \psi(x) = \frac{1}{c} \int_{x_0}^{x} g(y)\, dy - A\,, \tag{IV.1.10}$$

where x_0 and A are arbitrary constants, and therefore,

$$\begin{aligned} \phi(x) &= \frac{1}{2}(f(x) + A) - \frac{1}{2c} \int_{x_0}^{x} g(y)\,dy \\ \psi(x) &= \frac{1}{2}(f(x) - A) + \frac{1}{2c} \int_{x_0}^{x} g(y)\,dy\,. \end{aligned} \tag{IV.1.11}$$

Substituting x for $x + ct$ in ϕ and x for $x - ct$ in ψ, the final expressions of the displacement and velocity,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\,dy \\ \frac{\partial u}{\partial t}(x,t) &= \frac{c}{2}\Big(-f'(x-ct) + f'(x+ct)\Big) + \frac{1}{2}\Big(g(x-ct) + g(x+ct)\Big)\,, \end{aligned} \tag{IV.1.12}$$

highlight the influences of an initial displacement $f(x)$ and of an initial velocity $g(x)$.

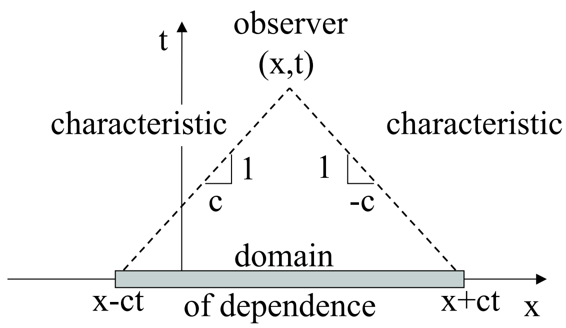


Figure IV.1 Sketch illustrating the notion of domain of dependence of the solution of the wave equation at a point (x, t).

IV.1.2.1 Domain of dependence

An observer sitting at point (x, t) sees two characteristics coming to him, $x - ct$ and $x + ct$ respectively. These characteristics bring the effects

- of an initial displacement f at $x - ct$ and $x + ct$ *only*;
- of an initial velocity g *all along the interval* $[x - ct, x + ct]$.

Furthermore, the velocity at point (x, t) is effected only by f' and g at $x - ct$ and $x + ct$. It is important to realize that the data outside this interval do not effect the solution at (x, t).

IV.1.2.2 Zone of influence

Conversely, it is also of interest to consider the domain of the (x, t)-plane that data at the point $(x_0, t = 0)$ influence. In fact, this domain is a triangular zone delimited by the characteristics $\xi = x_0 - ct$ and $\eta = x_0 + ct$.

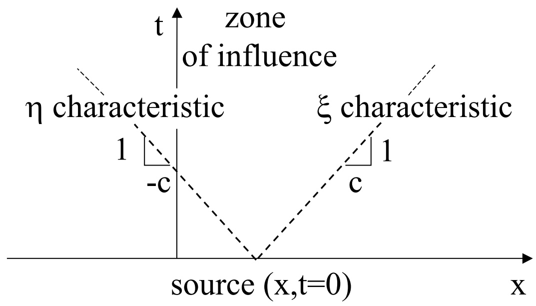


Figure IV.2 Sketch illustrating the notion of zone of influence of the initial data.

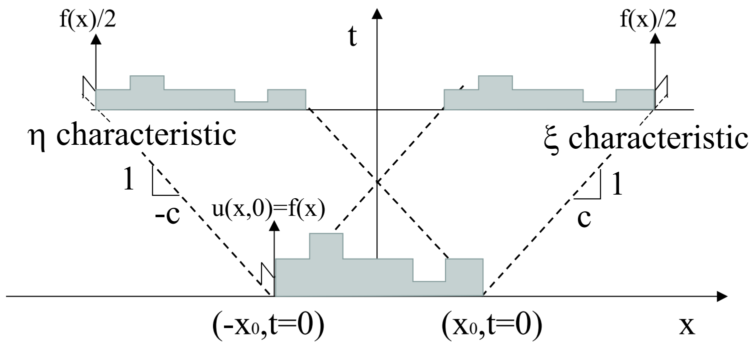


Figure IV.3 Sketch illustrating how an initial displacement is propagated form-invariant, but with half magnitude along each of the two characteristics.

IV.1.2.3 Effects of an initial displacement

The effect of an initial displacement can be illustrated by considering the special data,

$$f(x) = \begin{cases} f_0(x) & |x| \le x_0 \\ 0\,, & |x| > x_0, \end{cases} \quad ; \quad g(x) = 0, \quad -\infty < x < \infty\,. \tag{IV.1.13}$$

The solution (IV.1.12),

$$\begin{aligned} u(x,t) &= \frac{1}{2}\Big(f_0(x-ct) + f_0(x+ct)\Big) \\ \frac{\partial u}{\partial t}(x,t) &= \frac{c}{2}\Big(-f_0'(x-ct) + f_0'(x+ct)\Big)\,, \end{aligned} \tag{IV.1.14}$$

indicates that that the initial disturbance $f_0(x)$ propagates *without alteration* along the two characteristics $\xi = x - ct$ and $\eta = x + ct$, but scaled by a factor 1/2.

IV.1.2.4 Effects of an initial velocity

The effect of an initial velocity,

$$f(x) = 0, \quad -\infty < x < \infty; \quad g(x) = \begin{cases} g_0(x) & |x| \le x_0 \\ 0\,, & |x| > x_0, \end{cases} \tag{IV.1.15}$$

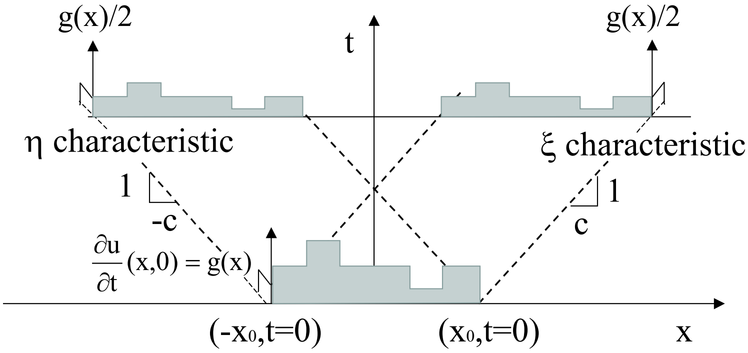


Figure IV.4 Sketch illustrating how an initial velocity is propagated form-invariant, but with half magnitude along each of the two characteristics.

on the displacement field can also be inspected via (IV.1.12),

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct} g_0(y)\,dy \\ \frac{\partial u}{\partial t}(x,t) &= \frac{1}{2}\Big(g_0(x-ct)+g_0(x+ct)\Big), \end{aligned} \tag{IV.1.16}$$

The effect on the velocity is simpler to address. In fact, this effect is similar to that of an initial displacement on the displacement field, as described above.

IV.1.3 The inhomogeneous wave equation

The additional effect of a volume source is considered in Exercise IV.1.

IV.1.4 A semi-infinite body. Reflection at boundaries

Thus far, we have been concerned with an infinite body. The idea was to avoid reflection of signals impinging boundaries located at finite distance.

With the basic presentation in mind, we can now address this phenomenon. This is the aim of Exercise IV.2.

IV.2 Conservation law and shock

Most field equations in engineering stem from balance statements. Matter or energy may be transported in space. Matter may undergo physical changes, like phase transform, aggregation, erosion $\cdots$. Energy may be used by various physical processes, or even change nature, from electrical or chemical turned mechanical.

Still in all these processes some entity is conserved, typically mass, momentum or energy. We explore here the basic mathematical structure of conservation laws, and the consequences in the solution of PDEs via the method of characteristics.

IV.2.1 Conservation law

A scalar (one-dimensional) conservation law is a partial differential equation of the form,

$$\frac{\partial u}{\partial t}+\frac{\partial q}{\partial x}=0,\quad \frac{\partial u}{\partial t}+\frac{dq}{du}\frac{\partial u}{\partial x}=0\,, \tag{IV.2.1}$$

where

- $u=u(x,t)$ is the primary unknown, representing for example, the density of particles along a line, or the density of vehicles along the segment of a road devoid of entrances and exits;
- $q(x,t)$ is the flux of particles, vehicles $\cdots$ crossing the position x at time t. This flux is linked to the primary unknown u, by a **constitutive relation** $q=q(u)$ that characterizes the flow.

Perhaps the simplest conservation law is

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\Big(\frac{1}{2}u^2\Big)=0,\quad \frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial x}=0\,. \tag{IV.2.2}$$

The proof of the conservation law goes as follows. Let us consider a segment $[a,b]$,

- along which all particles move with some non zero velocity;
- such that all particles that enter at $x=a$ exit at $x=b$, and conversely.

One defines

- the particle density as $u(x,t)$= nb of particles per unit length;
- the flux as $q(x,t)$= nb. of particles crossing the position x per unit time.

The conservation of particles in the section $[a,b]$ can be stated as follows: the variation of the nb of particles in this section is equal to the difference between the fluxes at a and b:

$$\frac{d}{dt}\int_a^b u(x,t)\,dx+q(b,t)-q(a,t)=0\,. \tag{IV.2.3}$$

Given that a and b are fixed positions, this relation can be rewritten,

$$\int_a^b \frac{\partial u(x,t)}{\partial t}+\frac{\partial q(x,t)}{\partial x}\,dx=0\,, \tag{IV.2.4}$$

whence the partial differential relation (IV.2.1), given that a and b are arbitrary.

IV.2.2 Shock and the jump relation

IV.2.2.1 Under and overdetermined characteristic network

Let us first consider a simple initial value problem (IVP), motivated by the sketch displayed in Fig. IV.5. We would like to solve the following problem for $u=u(x,t)$,

$$\begin{aligned}
&\text{(FE)} \quad \frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial x} &=&\ 0,\quad -\infty<x<\infty,\ t>0\\
&\text{(IC)} \quad\quad u(x,0) &=&\ \begin{cases} A, & x<0\\ B, & x\geq 0\,.\end{cases}
\end{aligned} \tag{IV.2.5}$$

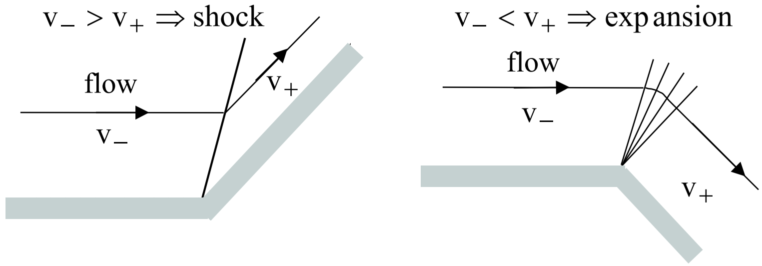


Figure IV.5 Qualitative sketch illustrating the shock-expansion theory. The flow velocity may decrease over a concave corner, or increase over a convex corner.

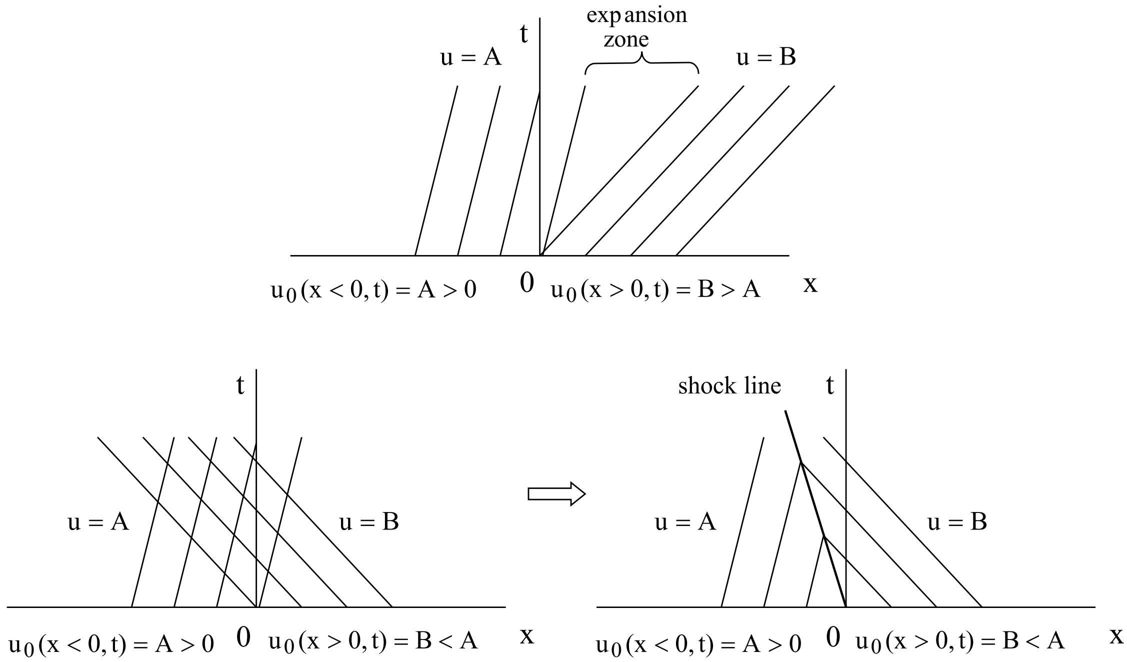


Figure IV.6 If the signal travels slower at the rear than at the front ($A < B$), the characteristic network is under-determined. Conversely, if the signal travels faster at the rear than in front ($A > B$), the characteristic network is over determined: the tentative network that displays intersecting characteristics, has to be modified to show a discontinuity line (curve).

Along a standard presentation, we would like the two relations,

$$\begin{aligned} 0 &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\, u \\ du &= \frac{\partial u}{\partial t}\, dt + \frac{\partial u}{\partial x}\, dx\,, \end{aligned} \tag{IV.2.6}$$

to be identical. Therefore, we should have simultaneously $dx/dt = u$, and $du = 0$. In other words, the characteristic curves are

$$\frac{dx}{dt} = u = \text{constant}\,. \tag{IV.2.7}$$

The construction of the characteristic network starts from the x-axis, Fig. IV.6.

Clearly the properties of the network depends on the relative values of A and B:

- for $A < B$, the characteristic network is underdetermined. There is a fan in which no characteristic exists. The signal emanating from points $(x < 0, t = 0)$ travels at a speed A slower than the signal emanating from points $(x > 0, t = 0)$;
- for $A > B$, the characteristic network is overdetermined, i.e. the characteristics would tend to intersect. Indeed, the signal emanating from points $(x < 0, t = 0)$ travels at a speed A greater than the signal emanating from points $(x > 0, t = 0)$. However the characteristics can not cross because the solution, e.g. a mass density, would be multi-valued.

IV.2.2.2 The jump relation across a shock

We now return to the conservation law for the unknown $u = (x, t)$ where $q = q(u)$ is seen as the flux,

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0\,. \tag{IV.2.8}$$

Let the symbol $[\![\cdot]\!]$ denote the jump across the shock,

$$[\![\cdot]\!] = (\cdot)_+ - (\cdot)_-\,, \tag{IV.2.9}$$

the symbol plus and minus indicating points right in front and right behind the shock. Of course the exact definition of the jump operator depends on what we call front and back, but the jump relation below does not.

The speed of propagation of the shock,

$$\frac{dX_s(t)}{dt} = \frac{[\![q]\!]}{[\![u]\!]} = \frac{q_+ - q_-}{u_+ - u_-}\,, \tag{IV.2.10}$$

depends on the jumps of the unknown $[\![u]\!]$ and flux $[\![q]\!]$ across the shock line.

The proof of this so-called **jump relation** begins by integration of the conservation law between two lagrangian positions $X_1 = X_1(t)$ and $X_2 = X_2(t)$,

$$\int_{X_1(t)}^{X_2(t)} \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x}\, dx = 0\,. \tag{IV.2.11}$$

This relation is further transformed using the standard formula that gives the derivative of an integral with variable and differentiable bounds,

$$\begin{aligned} &\frac{d}{dt}\int_{X_1(t)}^{X_2(t)} u(x,t)\, dx \\ &= \int_{X_1(t)}^{X_2(t)} \frac{\partial u(x,t)}{\partial t}\, dx + \frac{dX_2(t)}{dt}\, u(X_2(t),t) - \frac{dX_1(t)}{dt}\, u(X_1(t),t)\,. \end{aligned} \tag{IV.2.12}$$

Thus (IV.2.11) becomes

$$\frac{d}{dt}\int_{X_1}^{X_2} u(x,t)\, dx - \frac{dX_2}{dt}\, u(X_2,t) + \frac{dX_1}{dt}\, u(X_1,t) + q(X_2,t) - q(X_1,t) = 0\,. \tag{IV.2.13}$$

Finally we account for the fact that the shock has an infinitesimal width, so that, X_s being a point on the shock line at time t, letting X_1 tend to X_{s-} and X_2 tend to X_{s+}, we get

$$-\frac{dX_s}{dt}\, u_+ + \frac{dX_s}{dt}\, u_- + q_+ - q_- = 0 \quad \square\,. \tag{IV.2.14}$$

Remark: the shock relation applied to the mass conservation

Conservation of mass corresponds to $u = \rho$ mass density and to $q = \rho v$ momentum. The jump relation can be transformed to the standard relation that involves the Lagrangian speed of propagation of the shock line,

$$[\![\rho\,(\frac{dX_s}{dt} - v)]\!] = 0\,. \tag{IV.2.15}$$

IV.2.2.3 The entropy condition

Exercises IV.4 and IV.5 present examples of under and over determined characteristic networks. Exercise IV.4 indicates how to construct a solution in absence of characteristics. The underlying construction is in agreement with the entropy condition,

$$\frac{dq_-}{du} > \frac{dX_s(t)}{dt} > \frac{dq_+}{du}\,. \tag{IV.2.16}$$

Remark: on the intrinsic form of the conservation law

Consider the two distinct conservation laws, written in integral (intrinsic) form,

$$\frac{d}{dt}\int_a^b u(x,t)\,dx + \frac{1}{2}u^2(b,t) - \frac{1}{2}u^2(a,t) = 0\,, \tag{IV.2.17}$$

and

$$\frac{d}{dt}\int_a^b u^2(x,t)\,dx + \frac{2}{3}u^3(b,t) - \frac{2}{3}u^3(a,t) = 0\,, \tag{IV.2.18}$$

whose local forms (partial differential equations) are respectively,

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0\,, \tag{IV.2.19}$$

and

$$2\,u\,\Big(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x}\Big) = 0\,. \tag{IV.2.20}$$

As a conclusion, two distinct conservation laws may have identical local form. An issue arises in presence of a shock: on the shock line, the relation to be accounted for is the jump relation, and no longer the local relation. Consequently, the original (intrinsic) flux corresponding to the physical problem to be solved should be known and referred to.

IV.3 Guidelines to solve PDEs via the method of characteristics

As already alluded for, the method of characteristics to solve PDEs is a bit tricky. The method is quite general. As a consequence, a number of decisions has to be taken. This concerns in particular the choice of the curvilinear system. Any inappropriate choice may be bound to failure. Some basic notions are listed below. They should be complemented by exercises.

We would like to find the solution to the quasi-linear partial differential equation for $u = u(x,y)$,

$$a(x,y,u)\,\frac{\partial u}{\partial x} + b(x,y,u)\,\frac{\partial u}{\partial y} = c(x,y,u)\ , \tag{IV.3.1}$$

where the functions a, b and c are sufficiently smooth, with the boundary data,

$$u = u_0(s), \quad \text{along } I_0 : \left\{ \begin{array}{l} x = F(s) \\ y = G(s) \end{array} \right. . \tag{IV.3.2}$$

I_0 should not be a characteristic: if it is differentiable, this implies,

$$\frac{F'(s)}{G'(s)} \neq \frac{a(F(s),G(s),u_0(s))}{b(F(s),G(s),u_0(s))}\,. \tag{IV.3.3}$$

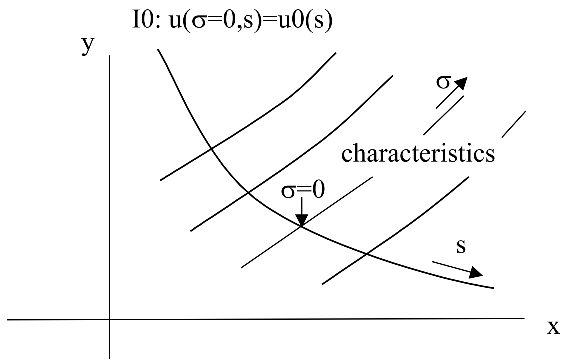


Figure IV.7 Data curve I_0, characteristic network and curvilinear coordinate σ associated with any characteristic and s associated with the curve I_0.

The method proceeds as follows. The curvilinear abscissa along the curve I_0 is s, Fig. IV.7. The curvilinear abscissa σ along a characteristic is arbitrarily, but conveniently, set to 0 on the curve I_0.

We would like the two relations,

$$\begin{aligned} c &= \frac{\partial u}{\partial x} a + \frac{\partial u}{\partial y} b \\ \frac{du}{d\sigma} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \sigma} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \sigma}\,, \end{aligned} \tag{IV.3.4}$$

to be identical. Therefore,

$$\frac{\partial x}{\partial \sigma} = a, \quad \frac{\partial y}{\partial \sigma} = b, \quad \frac{du}{d\sigma} = c\,, \tag{IV.3.5}$$

and, switching from the coordinates (x, y) to the coordinates (s, σ),

$$x(\sigma = 0, s) = F(s), \quad y(\sigma = 0, s) = G(s), \quad u(\sigma = 0, s) = u_0(s)\,. \tag{IV.3.6}$$

The solution is sought in the format,

$$x = x(\sigma, s), \quad y = y(\sigma, s), \quad u = u(\sigma, s)\,. \tag{IV.3.7}$$

The underlying idea is to fix s, so that (IV.3.4) becomes an ordinary differential equation (ODE). In other words, along each characteristic, (IV.3.4) is an ODE.

The system can be inverted into

$$\sigma = \sigma(x, y), \quad s = s(x, y)\,, \tag{IV.3.8}$$

if the determinant of the associated jacobian matrix does not vanish,

$$\frac{\partial(x,y)}{\partial(\sigma,s)} = \frac{\partial x}{\partial \sigma}\frac{\partial y}{\partial s} - \frac{\partial y}{\partial \sigma}\frac{\partial x}{\partial s} = a\,G'(s) - b\,F'(s) \neq 0\,. \tag{IV.3.9}$$

Exercise IV.1: **Inhomogeneous waves over an infinite domain**.

Consider the initial value problem governing the axial displacement $u(x,t)$,

$$\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial^2 u}{\partial t^2} - c^2\,\frac{\partial^2 u}{\partial x^2} = h(x,t), \quad t>0,\ x \in]-\infty,\infty[;\\
&\text{(IC) initial conditions} \quad u(x,0) = f(x); \quad \frac{\partial u}{\partial t}(x,0) = g(x)\,;\\
&\text{(BC) boundary conditions} \quad u(x\to\pm\infty,t) = 0\,,
\end{aligned} \tag{1}$$

in an infinite elastic bar,

$$-\infty \quad \cdots\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\cdots \quad +\infty \tag{2}$$

subject to prescribed initial displacement and velocity fields, $f = f(x)$ and $g = g(x)$ respectively. Here c is speed of elastic waves.

Show that the solution reads,

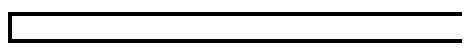
$$u(x,t) = \frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c\,(t-\tau)}^{x+c\,(t-\tau)} h(y,\tau)\ dy\ d\tau\,. \tag{3}$$

As an alternative to the Fourier transform used in Exercise II.7, exploit the method of characteristics.

Exercise IV.2: **Reflection of waves at a fixed boundary. The method of images**.

Sect. IV.1.1 has considered the propagation of initial disturbances in an infinite bar. The idea was to avoid reflections of the signal that was bounded to complicate the initial exposition.

We turn here to the case of a semi-infinite bar,

0  $\cdots \quad \infty$ (1)

whose boundary $x = 0$ is fixed. The conditions (IV.1.5) modify to

$$
\begin{aligned}
&(\mathrm{CL})_1 \quad u(x=0, t=0) = 0; \quad u(x \to +\infty, t=0) = 0; \\
&(\mathrm{CL})_2 \quad \frac{\partial u}{\partial t}(x=0, t=0) = 0; \quad \frac{\partial u}{\partial t}(x \to +\infty, t=0) = 0\,.
\end{aligned}
\tag{2}
$$

The method of images consist

- in thinking of a mirror bar over $]-\infty, 0[$;
- in complementing the initial data (IV.1.4) over the real bar by the data,

$$
f(x) = -f(-x); \quad g(x) = -g(-x); \quad x < 0\,.
\tag{3}
$$

Show that the displacement and velocity fields (IV.1.12) for the infinite bar become,

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\Big(\mathrm{sgn(x-ct)}\, f(|x-ct|) + f(x+ct)\Big) + \frac{1}{2c}\int_{|x-ct|}^{x+ct} g(y)\, dy \\
\frac{\partial u}{\partial t}(x,t) &= \frac{c}{2}\Big(-f'(|x-ct|) + f'(x+ct)\Big) + \frac{1}{2}\Big(\mathrm{sgn(x-ct)}\, g(|x-ct|) + g(x+ct)\Big)\,,
\end{aligned}
\tag{4}
$$

for the semi-infinite bar extending over $[0, +\infty[$.

Exercise IV.3: **A first order quasi-linear partial differential equation with boundary conditions**.

Find the solution to the partial differential equation for $u = u(x, y)$,

$$x\,\frac{\partial u}{\partial x} + y\,u\,\frac{\partial u}{\partial y} + x\,y = 0, \quad x > 0,\ y > 0\,, \tag{1}$$

with the boundary data,

$$u = 5, \quad \text{along } I_0 : x\,y = 1,\ x > 0\,. \tag{2}$$

Solution:

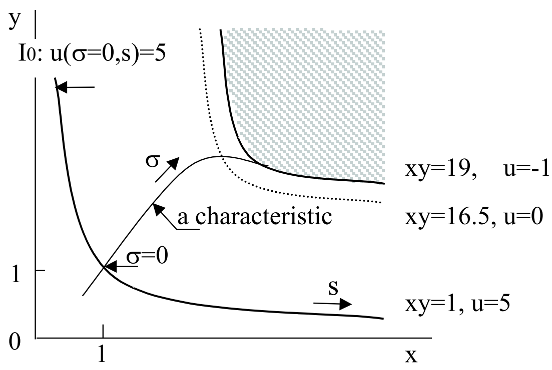


Figure IV.8 Curvilinear coordinates associated with the boundary value problem.

One can choose the curvilinear abscissa of the curve I_0 to be $s = x$, Fig. IV.8. The curvilinear abscissa σ along a characteristic is arbitrarily, but conveniently, set to 0 on the curve I_0.

Along the standard presentation, we would like the two relations,

$$\begin{aligned} -x\,y &= \frac{\partial u}{\partial x}\,x &+& \frac{\partial u}{\partial y}\,y\,u \\ \frac{du}{d\sigma} &= \frac{\partial u}{\partial x}\,\frac{\partial x}{\partial \sigma} &+& \frac{\partial u}{\partial y}\,\frac{\partial y}{\partial \sigma}\,, \end{aligned} \tag{3}$$

to be identical. Therefore, the characteristic curves are defined by the relations $dy/dx = y\,u/x$, and

$$\frac{\partial x}{\partial \sigma} = x, \quad \frac{\partial y}{\partial \sigma} = y\,u, \quad \frac{\partial u}{\partial \sigma} = -x\,y\,, \tag{4}$$

and, switching from the coordinates (x, y) to the coordinates (s, σ),

$$x(\sigma = 0, s) = s, \quad y(\sigma = 0, s) = \frac{1}{s}, \quad u(\sigma = 0, s) = 5,\ s > 0\,. \tag{5}$$

To integrate (4), we note

$$\begin{aligned} \frac{\partial(x\,y)}{\partial\sigma} &= \underbrace{\frac{\partial x}{\partial\sigma}}_{=x,\ (4)_1} y + x \underbrace{\frac{\partial y}{\partial\sigma}}_{=yu,\ (4)_2} \\ &= (1+u)\ \underbrace{x\,y}_{(4)_3} \\ &= -(1+u)\,\frac{\partial u}{\partial\sigma} \\ &= -\frac{\partial}{\partial\sigma}(u + \frac{u^2}{2})\,. \end{aligned} \tag{6}$$

Therefore,

$$xy = -u - \frac{u^2}{2} + \phi(s)\,. \tag{7}$$

The function $\phi(s)$ is fixed by the boundary condition (2),

$$1 = -5 - \frac{5^2}{2} + \phi(s) \quad \Rightarrow \quad \phi(s) = \frac{37}{2}\,. \tag{8}$$

In summary, the solution of (7) which also satisfies the boundary condition along I_0, is

$$u(x,y) = -1 + \sqrt{38 - 2\,x\,y}, \quad x\,y < 19\,. \tag{9}$$

Exercise IV.4: **An IBVP with an expansion zone**.

1. Consider the first order partial differential equation for the unknown $u = u(x,t)$,

$$\frac{\partial u}{\partial t} + a(u)\,\frac{\partial u}{\partial x} = 0\,, \tag{1}$$

where a is a function of u. Show that the solutions of the form $u(x,t) = f(x/t)$ are the constants and the generalized inverses of a, that is, the functions such the composition of a and f is the identity function, $a \circ f = I$.

2. Solve the IBVP for $u = u(x,t)$,

$$\begin{array}{lll} \text{(FE)} & \dfrac{\partial u}{\partial t} + e^u\,\dfrac{\partial u}{\partial x} = 0, & x > 0,\ t > 0 \\ \text{(IC)} & u(x,0) = 2, & x > 0 \\ \text{(BC)} & u(0,t) = 1, & t > 0\,. \end{array} \tag{2}$$

Solution:

1. If $u(x,t) = f(x/t)$, then

$$\frac{\partial u}{\partial t} = -\frac{x}{t^2}\,f', \quad \frac{\partial u}{\partial x} = \frac{1}{t}\,f', \tag{3}$$

and therefore,

$$\frac{\partial u}{\partial t} + a\,\frac{\partial u}{\partial x} = \frac{f'}{t}\Big(-\frac{x}{t} + a\Big), \tag{4}$$

whence, either f is constant or

$$\frac{x}{t} = a = a(u) = a\Big(f(\frac{x}{t})\Big) \quad \Rightarrow \quad a \circ f = I\,. \tag{5}$$

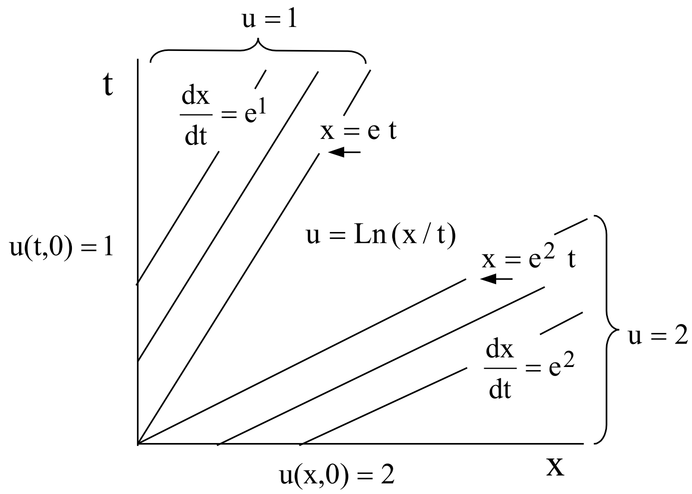


Figure IV.9 In the central fan where no characteristic exists, the solution is built heuristically. It connects continuously with the two zones where the solution carried out by the characteristics is constant.

2. Along the standard presentation, we would like the two relations,

$$\begin{aligned} 0 &= \frac{\partial u}{\partial t} &+& \frac{\partial u}{\partial x}\, e^{u} \\ du &= \frac{\partial u}{\partial t}\, dt &+& \frac{\partial u}{\partial x}\, dx\,, \end{aligned} \tag{6}$$

to be identical. Therefore, we should have simultaneously $dx/dt = e^u$, and $du = 0$. In other words, the characteristic curves are

$$\frac{dx}{dt} = e^u = \text{constant}\,. \tag{7}$$

The construction of the characteristic network starts from the axes, Fig. IV.9.

There is no characteristic curve in a central fan. Still, we have a family of solutions via question 1. Since here the function a is the exponential, the inverse is the Logarithm. Continuity at the boundaries of the central fan is ensured simply by taking $u(x,t) = \mathrm{Ln}(x/t)$.

In summary,

$$u(x,t) = \begin{cases} 1, & 0 < x \le e\,t \\ \mathrm{Ln}(x/t), & e\,t \le x \le e^2\,t \\ 2, & e^2\,t \le x\,. \end{cases} \tag{8}$$

Note that we have not proved the uniqueness of the solution in the central fan. On the other hand, we can eliminate a jump from 1 to 0 along the putative shock line $X_s = \frac{1}{2}\,(1+0)\,t$, because this shock would not satisfy the entropy condition (IV.2.16).

Exercise IV.5: **An initial value problem (IVP) with a shock**.
Solve the initial value problem (IVP) for $u = u(x,t)$,

$$\begin{aligned} \text{(FE)} \quad \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} &= 0, \quad -\infty < x < \infty,\ t > 0 \\ \text{(IC)} \qquad u(x,0) &= \begin{cases} 1, & x \leq 0 \\ 1-x, & 0 < x < 1 \\ 0, & 1 \leq x\,. \end{cases} \end{aligned} \tag{1}$$

Solution:
Along the standard presentation, we would like the two relations,

$$\begin{aligned} 0 &= \frac{\partial u}{\partial t} \quad + \frac{\partial u}{\partial x}\,u \\ du &= \frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dx\,, \end{aligned} \tag{2}$$

to be identical. Therefore, we should have simultaneously $dx/dt = u$, and $du = 0$. In other words, the characteristic curves are

$$\frac{dx}{dt} = u = \text{constant}\,. \tag{3}$$

The construction of the characteristic network starts from the x-axis, Fig. IV.10.

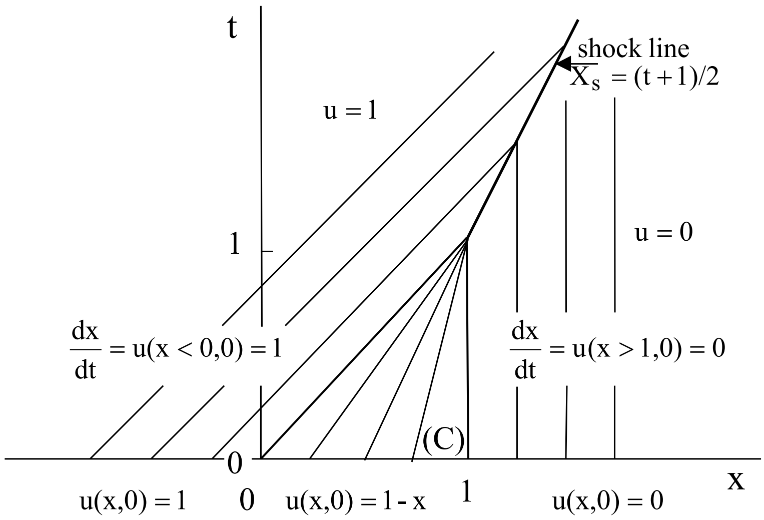


Figure IV.10 A shock develops to accommodate a faster information coming from behind. The shock line has equation $t = 2\,X_s - 1$, for $X_s > 1$. Elsewhere the solution is continuous.

The characteristics emanating from the x-axis for $x < 0$ carry the solution $u(x,t) = 1$, while the characteristics emanating from the x-axis for $x > 1$ carry the solution $u(x,t) = 0$.

However, we clearly have a problem, Fig. IV.10, because the above description implies that the characteristics cross each other, which is impossible. Consequently there is shock.

Let us first re-write the field equation as a conservation law,

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\Big(\frac{u^2}{2}\Big) = 0\,, \tag{4}$$

so as to identify the flux $q = q(u) = u^2/2$. The jump relation (IV.2.10) provides the speed of propagation of the shock line,

$$\frac{dX_s}{dt} = \frac{q_+ - q_-}{u_+ - u_-} = \frac{1}{2}\,(u_+ + u_-) = \frac{1}{2}\,, \tag{5}$$

the subscripts + and - denoting the two sides of the shock line. Therefore $X_s = t/2 +$ constant. The later constant is fixed by insisting that the point $(1, 1)$ belongs to the shock line. Therefore, the shock line is the semi-infinite segment,

$$X_s = \frac{t}{2} + \frac{1}{2}, \quad x \geq 1,\ t \geq 1\,. \tag{6}$$

To the left of the shock line, i.e. $x < (t + 1)/2$, the solution u is equal to 1, while it is equal to 0 to the right.

In the central fan (C), the slope of the characteristics dx/dt, which we know is constant, is equal to $u(x, 0) = 1 - x$. The tentative function,

$$u(x,t) = \frac{1-x}{1-t}, \quad x < 1,\ t < 1\,, \tag{7}$$

satisfies the field equation, has the proper slope at $t = 0$, and hence at any $t < 1$ since the slope is constant along characteristics, and fits continuously with the left and right characteristic networks.

Finally, note that the shock satisfies the entropy condition (IV.2.16).

Exercise IV.6: **A sudden surge in a river of Southern France**.

The height H and the horizontal velocity of water v in a long river, with a quasi horizontal bed, are governed by the equations of balance of mass and balance of horizontal momentum (here $g \sim 10$ m/s^2 is the gravitational acceleration):

$$\begin{aligned} &\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\,v) = 0, \\ &\frac{\partial}{\partial t}(Hv) + \frac{\partial}{\partial x}(H\,v^2 + \frac{g}{2}\,H^2) = 0\,. \end{aligned} \tag{1}$$

The horizontal velocity of water is $v_+ = 2\,\mathrm{m/s}$ and the height is $H_+ = 1\,\mathrm{m}$. A time $t = 0$, the height becomes suddenly equal to $H_- = 2\,\mathrm{m}$ upstream $x \leq 0$, and it keeps that value at later times $t > 0$. Neglecting frictional resistance, bed slope, the local physical effects in the neighborhood of the shock, deduce the horizontal velocity v_- of water behind the shock, and the speed of displacement of the shock dX_s/dt.

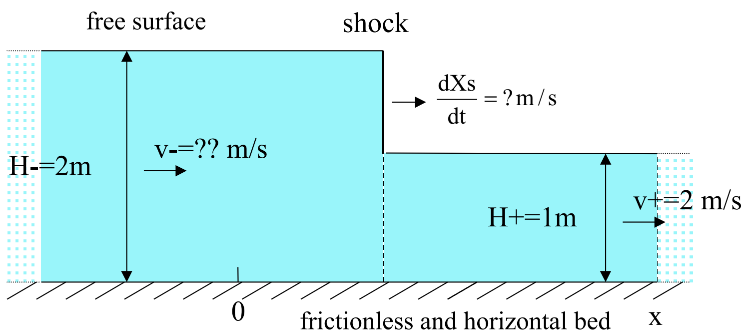


Figure IV.11 A shock develops to accommodate a faster information coming from upstream.

N.B. If the river bed is inclined downward with an angle $\theta > 0$, and if the coefficient of friction f is non zero, the rhs of the balance of momentum should be changed to $g\,H \sin\theta - f\,v^2$.

Solution:

The jump relation (IV.2.10) is applied to the two conservation equations,

$$\frac{dX_s}{dt} = \frac{(H\,v)_+ - (H\,v)_-}{H_+ - H_-} = \frac{(H\,v^2 + \frac{1}{2}g\,H^2)_+ - (H\,v^2 + \frac{1}{2}g\,H^2)_-}{(H\,v)_+ - (H\,v)_-}\,, \tag{2}$$

the subscripts + and - denoting the two sides of the shock line. Solving this equation for v_- yields,

$$\frac{v_- - v_+}{H_- - H_+} = \epsilon\sqrt{\frac{g}{2}\,(\frac{1}{H_+} + \frac{1}{H_-})}\,, \tag{3}$$

that is,

$$v_- = v_+ + \epsilon\,(H_- - H_+)\sqrt{\frac{g}{2}\,(\frac{1}{H_+} + \frac{1}{H_-})} \sim 4.74\,\mathrm{m/s}\,, \tag{4}$$

from which follows the speed of propagation of the shock,

$$\frac{dX_s}{dt} = v_+ + \epsilon\,H_-\sqrt{\frac{g}{2}\,(\frac{1}{H_+} + \frac{1}{H_-})} \sim 7.48\,\mathrm{m/s}\,. \tag{5}$$

In fact, there are two solutions to the problem as indicated by by $\epsilon = \pm 1$. The choice $\epsilon = +1$ is dictated by the entropy condition that implies that the upstream flow should be larger than the downstream flow.

Exercise IV.7: **Implicit solution to an initial value problem (IVP). Simple waves**.

The conservation law $\partial u/\partial t + \partial q(u)/\partial x = 0$ may also be written $\partial u/\partial t + a(u)\,\partial u/\partial x = 0$, with $a(u) = dq/du$. Let us assume $a(u) > 0$.

1. Show that the solution to the initial value problem (IVP) for $u = u(x,t)$,

$$\begin{array}{lrcl} \text{(FE)} & \dfrac{\partial u}{\partial t} + a(u)\,\dfrac{\partial u}{\partial x} & = & 0, \quad -\infty < x < \infty,\ t > 0 \\ \text{(IC)} & u(x,0) & = & u_0(x), \quad -\infty < x < \infty, \end{array} \tag{1}$$

is given implicitly under the format,

$$u = u_0(x - a(u)\,t)\,, \tag{2}$$

if

$$1 + u_0'(x - a(u)\,t)\,a'(u)\,t \neq 0\,. \tag{3}$$

Such a solution is a *forward simple wave*. Indeed, it travels at increasing x. Simple waves are waves that get distorted because their speed depends on the solution u.

2. Define similarly *backward* simple waves.

Solution:

1. The curve I_0 on which the data are given is the x-axis, so that one can choose the curvilinear abscissa of the curve I_0 to be $s = x$. The curvilinear abscissa σ along a characteristic is arbitrarily, but conveniently, set to 0 on the curve I_0.

Along the standard presentation, we would like the two relations,

$$\begin{array}{rcccc} 0 & = & \dfrac{\partial u}{\partial t} & + & \dfrac{\partial u}{\partial x}\,a(u) \\ \dfrac{du}{d\sigma} & = & \dfrac{\partial u}{\partial t}\,\dfrac{\partial t}{\partial \sigma} & + & \dfrac{\partial u}{\partial x}\,\dfrac{\partial x}{\partial \sigma}\,, \end{array} \tag{4}$$

to be identical. Therefore, the characteristic curves are defined by the relations $dx/d\sigma = a(u)$ and $du = 0$, that is, the solution u is constant along a characteristic,

$$\frac{\partial t}{\partial \sigma} = 1, \quad \frac{\partial x}{\partial \sigma} = a(u), \quad \frac{\partial u}{\partial \sigma} = 0\,, \tag{5}$$

and, switching from the coordinates (x,t) to the coordinates (s,σ),

$$t(\sigma = 0, s) = 0, \quad x(\sigma = 0, s) = s, \quad u(\sigma = 0, s) = u_0(s)\,. \tag{6}$$

The construction of the characteristic network starts from the x-axis, Fig. IV.12.

From $(5)_3$ and $(6)_3$ results

$$u(\sigma, s) = u(\sigma = 0, s) = u_0(s)\,. \tag{7}$$

Relations $(5)_1$ and $(6)_1$ imply,

$$t = \sigma + \phi(s) \overset{(6)_1}{=} \sigma\,. \tag{8}$$

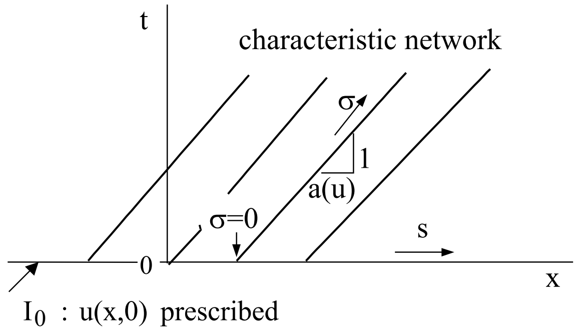


Figure IV.12 Curvilinear coordinates associated with an initial value problem.

In addition, relation $(5)_2$ can be integrated with help of $(6)_2$ and (7),

$$x = \sigma\, a(u_0) + \psi(s) \overset{(6)_2}{=} \sigma\, a(u_0) + s\,. \tag{9}$$

Finally, collecting these relations yields an implicit equation for u,

$$u \overset{(7)}{=} u_0(s) \overset{(9)}{=} u_0\Big(x - \sigma\, a(u)\Big) \overset{(8)}{=} u_0\Big(x - t\, a(u)\Big)\,. \tag{10}$$

Let us now try to solve this equation by differentiation,

$$du = \Big(dx - dt\, a(u) - t\, a'(u)\, du\Big)\, u_0'\,, \tag{11}$$

from which we can extract du,

$$(1 + t\, a'(u)\, u_0')\, du = (dx - dt\, a(u))\, u_0'\,, \tag{12}$$

only under the condition (3), which is in fact a particular form of the so-called theorem of implicit functions.

2. By deduction, backward simple waves are defined implicitly by the relation,

$$u = u_0(x + a(-u)\, t)\,, \tag{13}$$

and obey the PDE,

$$\frac{\partial u}{\partial t} - a(-u)\, \frac{\partial u}{\partial x} = 0\,. \tag{14}$$

Exercise IV.8: **Transient flow of a compressible fluid at constant pressure: a weakly coupled problem**.

Under adiabatic conditions, the velocity $u(x,t)$, mass density $\rho(x,t)$ and internal energy per unit volume $e(x,t)$ during the one-dimensional flow of a compressible fluid at constant pressure p are governed by the three coupled nonlinear partial differential equations, for $-\infty < x < \infty,\ t > 0$,

$$\begin{aligned} \text{momentum equation}: &\quad \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0 \\ \text{mass conservation}: &\quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \\ \text{energy equation}: &\quad \frac{\partial e}{\partial t} + \frac{\partial}{\partial x}(ue) + p\,\frac{\partial u}{\partial x} = 0\,, \end{aligned} \tag{1}$$

subject to the initial data:

$$u(x,0) = u_0(x); \quad \rho(x,0) = \rho_0(x); \quad e(x,0) = e_0(x), \quad -\infty < x < \infty\,. \tag{2}$$

The function $u_0(x)$ is assumed to be differentiable and the functions $\rho_0(x)$ and $e_0(x)$ to be continuous.

1. Solve this system for the three unknowns velocity $u(x,t)$, density $\rho(x,t)$ and internal energy $e(x,t)$.

2. The above three equations have been worked out and organized so as to be brought into an easily solvable weakly coupled system. Show that this system of equations actually derives from the conservation laws of mass, momentum, and total (internal plus kinetic) energy, namely in turn,

$$\begin{aligned} \text{mass conservation}: &\quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \\ \text{momentum balance}: &\quad \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(p + \rho\, u^2) = 0 \\ \text{energy conservation}: &\quad \frac{\partial}{\partial t}(e + \tfrac{1}{2}\rho\, u^2) + \frac{\partial}{\partial x}\Big(u\,(e + \tfrac{1}{2}\rho\, u^2 + p)\Big) = 0\,. \end{aligned} \tag{3}$$

Solution:

Note that the order in which we have written the three equations matters. It is important to recognize that the first equation is independent and involves the sole unknown u, while the two other equations involve two unknowns. Therefore, the analysis begins by the uncoupled equation.

1.1 The first equation is a particular case of Exercise IV.7, with $a(u) = u$. From (8) and (9) of this Exercise, the characteristics are

$$x - u\,t = s\,, \tag{4}$$

and the solution is given implicitly by the equation,

$$u = u_0(s) = u_0(x - u\,t)\,. \tag{5}$$

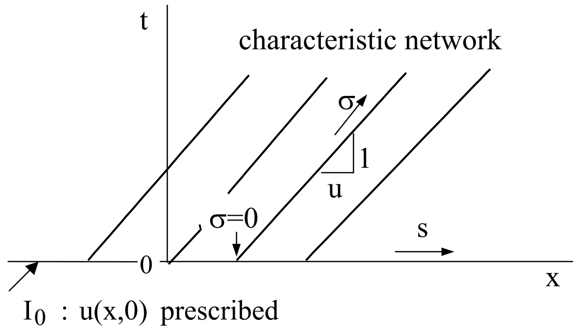


Figure IV.13 Curvilinear coordinates associated with the initial value problem.

1.2 The second equation may be re-written,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\, u = -\rho\, \frac{\partial u}{\partial x} \,. \tag{6}$$

The right hand side being known, this equation has the same characteristics defined by $dx/dt = u$ and $\sigma = t$, as the first one. Along a characteristic,

$$\frac{\partial \rho}{\partial t}\, \frac{\partial t}{\partial \sigma} + \frac{\partial \rho}{\partial x}\, \frac{\partial x}{\partial \sigma} = \frac{d\rho}{d\sigma} \,. \tag{7}$$

On comparing the last two equations,

$$\frac{\partial \rho}{\partial t} = \frac{d\rho}{d\sigma} = -\rho\, \frac{\partial u}{\partial x} \,. \tag{8}$$

Now, from (5),

$$\frac{\partial u}{\partial x} = u_0'(s) \left(1 - t\, \frac{\partial u}{\partial x}\right), \tag{9}$$

and therefore,

$$\frac{\partial u}{\partial x} = \frac{u_0'(s)}{1 + t\, u_0'(s)} \,. \tag{10}$$

Insertion of this relation in (8),

$$\frac{1}{\rho}\, \frac{d\rho}{d\sigma} = -\frac{u_0'(s)}{1 + \sigma\, u_0'(s)} \,, \tag{11}$$

and integration with respect to σ, accounting for the fact that σ and s are independent variables, yields

$$\rho(\sigma, s) = \frac{\rho_0(s)}{1 + \sigma\, u_0'(s)} \,. \tag{12}$$

Return to the coordinates (x, t) uses the relations $t = \sigma$ and $s = x - u\,t$.

1.3 The third equation can be recast in terms of a new unknown $E(x, t) = e(x, t) + p$, namely

$$\frac{\partial E}{\partial t} + \frac{\partial E}{\partial x}\, u = -E\, \frac{\partial u}{\partial x} \,, \tag{13}$$

with initial data $E_0(x) = E(x,0) = e(x,0) + p$. This equation is clearly identical in form to (6), and has therefore solution,

$$E(\sigma, s) = \frac{E_0(s)}{1 + \sigma\, u_0'(s)} \quad \Rightarrow \quad e(\sigma, s) = \frac{e_0(s) - p\,\sigma\, u_0'(s)}{1 + \sigma\, u_0'(s)}\,. \tag{14}$$

2. Just combine the three equations.

For those who want to know more.
1. As indicated above, the expressions of the energy equation assume adiabatic conditions. More generally, the energy equation is contributed by heat exchanges with the surroundings, heat sources and heat wells.
2. The set of equations (14) actually holds even for a space and time dependent pressure.

Exercise IV.9: **Traffic flow along a road segment devoid of entrances and exits**.

We have seen in Sect.IV.2.1 that the flux $q(x,t)$ in a conservation law for a density $u(x,t)$ is linked constitutively with this density, namely $q = q(u) = q(u(x,t))$. To illustrate the issue, let us consider the traffic of vehicles along a road segment devoid of entrances and exits. In absence of vehicles $u = 0$, the flux of course vanishes, $q(0) = 0$. On the other hand, one may admit that vehicles need some fluidity to move: at the maximum density u_m, bumper to bumper, the flux vanishes, $q(u_m) = 0$. Consequently the relation $q = q(u)$ can not be linear. The simplest possibility is perhaps,

$$\frac{q(u)}{q_m} = \begin{cases} 4\frac{u}{u_m}\Big(1 - \frac{u}{u_m}\Big), & 0 \leq \frac{u}{u_m} \leq 1 \\ 0, & 1 < \frac{u}{u_m} \end{cases} . \tag{1}$$

The speed of the traffic,

$$v = \frac{q}{u} = u_m \Big(1 - \frac{u}{u_m}\Big), \quad u_m \equiv 4\,\frac{q_m}{u_m}\,, \tag{2}$$

is therefore an affine function of the density.

1. Define the characteristics. Show Property (P): the slopes of the characteristics are constant and the density u is constant along a characteristic.

2. Consider the effects of a traffic light located at the position $x = 0$. The light has been red for some time. Behind the light, the density is maximum while there is no vehicle ahead. The light turned green at time $t = 0$. Thus the initial density is $u(x, t = 0) = u_m\,\mathcal{H}(-x)$. Obtain the density $u(x,t)$ at later times $t > 0$. Consider also the trajectory of a particular vehicle.

3. The light has been green for some time, and the density is uniform and smaller than u_m. At time $t = 0$, the light turns red, and the density becomes instantaneously maximum at $x = 0_-$. Describe the shock that propagates to the rear.

N.B. This exercise could be interpreted as well as describing a particle flow in a tube with opening and closing gates.

Exercise IV.10: **Transmission lines subject to initial data or boundary data**.

We return to the transmission line problem described in Exercise III.6. The equations governing the current $I(x,t)$ and potential $V(x,t)$ in a transmission line of axis x can be cast in the format of a linear system of two partial differential equations,

$$\begin{aligned} L\frac{\partial I}{\partial t} + \frac{\partial V}{\partial x} + R\,I &= 0 \\ C\frac{\partial V}{\partial t} + \frac{\partial I}{\partial x} + G\,V &= 0 \end{aligned} \tag{1}$$

1. For an infinite transmission line,

$$-\infty \quad \cdots \overline{\overline{\qquad\qquad\qquad}} \cdots \quad +\infty \tag{2}$$

solve the initial value problem, defined by the field equations (1) for $-\infty < x < \infty$, $t > 0$, subjected to the initial conditions,

$$I(x, t = 0) = I_0(x), \quad V(x, t = 0) = V_0(x), \quad -\infty < x < \infty\,. \tag{3}$$

Use the method of characteristics together with the normal form of the equations phrased in terms of alternative variables provided by equations (6) of Exercise III.6.

Obtain an integral solution in the general case, and an analytical solution for a distortionless line, $RC = LG$, for which the set of field equations uncouples. Highlight the parameters that characterize propagation and time decay.

2. Consider now a semi-infinite transmission line extending over $x > 0$,

$$0 \; \sqsubset\!\!\!\!\equiv\!\equiv \cdots \quad \infty \tag{4}$$

Solve the boundary value problem defined by the field equations (1) for $0 < x < \infty$, $t > 0$, subjected to the boundary conditions,

$$I(x = 0, t) = I_0(t), \quad V(x = 0, t) = V_0(t), \quad t > 0\,. \tag{5}$$

Restrict the analysis to a distortionless line, $RC = LG$.