Chapter II

Solving IBVPs with Fourier transforms

The exponential (complex) Fourier transform is well adapted to solve IBVPs in *infinite* bodies, while the real Fourier transforms are better suited to address IBVPs in *semi-infinite* bodies. The choice between the sine and cosine Fourier transforms will be shown to depend on the boundary conditions.

When it can be obtained, the response to a point load, so called-Green function, is instrumental to build the response to arbitrary loading. [1]

II.1 Exponential Fourier transform for diffusion problems

We consider a diffusion phenomenon in an infinite bar,

$$-\infty \quad \cdots \qquad \cdots \quad +\infty \tag{II.1.1}$$

aligned with the x-axis, and endowed with a diffusion coefficient $D > 0$. For definiteness, the unknown field $u = u(x,t)$ may be interpreted as a temperature. The initial temperature $u = u(x,0)$ is a known function of space. The radiation condition imposes the temperature to vanish at infinity. The governing equations are,

$$\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial u}{\partial t} - D\,\frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0,\ x \in]-\infty,\infty[;\\
&\text{(IC) initial condition} \quad u(x,0) = h(x), \quad x \in]-\infty,\infty[\,;\\
&\text{(RC) radiation conditions} \quad u(|x| \to \infty, t) = 0\,.
\end{aligned} \tag{II.1.2}$$

To motivate the use of the Fourier transform, we observe that the space variable x varies from $-\infty$ to ∞, and we choose the transform

$$u(x,t) \to U(\alpha,t) = \mathcal{F}\{u(x,t)\}(\alpha)\,, \tag{II.1.3}$$

and we admit that the operators Fourier transform and partial derivative in time commute,

$$\frac{\partial}{\partial t}\mathcal{F}\{u(x,t)\}(\alpha) = \mathcal{F}\{\frac{\partial}{\partial t}u(x,t)\}(\alpha)\,. \tag{II.1.4}$$

[1] Posted, November 29, 2008; updated, April 03, 2009

The field equation becomes a ODE (ordinary differential equation) for $U(\alpha,t)$ where the Fourier variable α is seen as a parameter,

$$\begin{aligned} &\text{(FE)} \quad \frac{dU}{dt}(\alpha,t) + D\,\alpha^2\,U(\alpha,t) = 0, \quad t>0, \\ &\text{(CI)} \quad U(\alpha,0) = H(\alpha)\,, \end{aligned} \tag{II.1.5}$$

that solves to

$$U(\alpha,t) = H(\alpha)\,e^{-\alpha^2\,D\,t}\,. \tag{II.1.6}$$

The inverse is

$$\begin{aligned} u(x,t) &= \frac{1}{2\,\pi}\int_{-\infty}^{\infty} e^{i\,\alpha\,x-\alpha^2\,D\,t}\,H(\alpha)\,d\alpha \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\infty} \cos\,(\alpha(x-\xi))\,e^{-\alpha^2\,D\,t}\,d\alpha\,h(\xi)\,d\xi\,. \end{aligned} \tag{II.1.7}$$

The imaginary part has disappeared due to the fact that the integrand is even with respect to the variable α. This expression may be simplified by the use of a Green function.

II.1.1 The Green function as the solution to a point source at the origin

For a point source at the origin,

$$\text{(IC)} \quad u_\delta(x,0) = h(x) = \delta(x)\,, \tag{II.1.8}$$

the solution u_δ can be expressed in explicit form,

$$u_\delta(x,t) = \frac{1}{\pi}\int_0^{\infty} \cos\,(\alpha\,x)\,e^{-\alpha^2\,D\,t}\,d\alpha = \frac{1}{2\sqrt{\pi\,D\,t}}\,\exp\Big(-\frac{x^2}{4\,D\,t}\Big)\,. \tag{II.1.9}$$

The proof of (II.1.9) is detailed in exercise II.6.

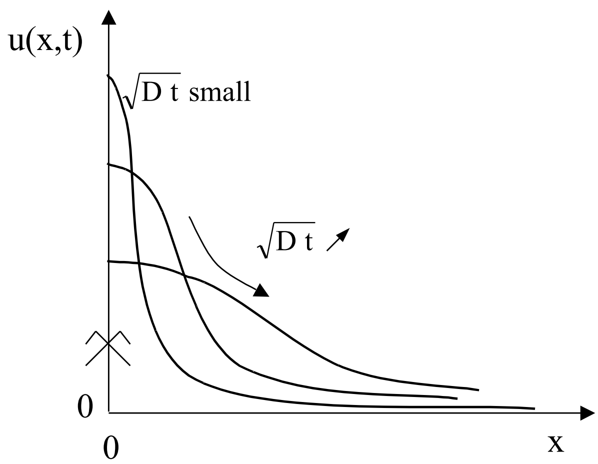

Figure II.1 An infinite bar subjected to a point heat source at the origin. Evolution of the spatial profile in time.

To check that the initial condition (IC) is satisfied, we use the fact, established in the Chapter on distributions, that, in the sense of distributions,

$$\lim_{\epsilon\to 0^+} \frac{1}{2\sqrt{\pi\,\epsilon}} \exp\Big(-\frac{x^2}{4\,\epsilon}\Big) = \delta(x)\,. \tag{II.1.10}$$

The formula (II.1.9) also yields the additional result, for $\kappa > 0$,

$$\mathcal{F}\Big\{\exp\Big(-\frac{x^2}{4\kappa}\Big)\Big\}(\alpha) = 2\sqrt{\pi\,\kappa}\exp(-\kappa\,\alpha^2), \quad \mathcal{F}^{-1}\{\exp(-\kappa\,\alpha^2)\}(x) = \frac{1}{2\sqrt{\pi\,\kappa}}\exp\Big(-\frac{x^2}{4\kappa}\Big)\,. \tag{II.1.11}$$

II.1.2 Point source at an arbitrary location

The solution to a point source $h(x) = \delta(x-\xi)$ at ξ is easily deduced as

$$\frac{1}{2\sqrt{\pi\,D\,t}} \exp\Big(-\frac{(x-\xi)^2}{4\,D\,t}\Big)\,. \tag{II.1.12}$$

II.1.3 Response to an arbitrary source via the Green function

For an arbitrary source, the solution may be built starting from the Green function u_δ. Indeed, since the Fourier transform of $\delta(x)$ is 1,

$$\begin{aligned} U_\delta(\alpha,t) &= e^{-\alpha^2\,D\,t}, \\ U(\alpha,t) &= H(\alpha)\,e^{-\alpha^2\,D\,t}, \end{aligned} \tag{II.1.13}$$

in view of (II.1.6), and therefore,

$$U(\alpha,t) = U_\delta(\alpha,t)\,H(\alpha)\,, \tag{II.1.14}$$

so that the inverse is a convolution product where the Green function appears as the kernel,

$$u(x,t) = \frac{1}{2\sqrt{\pi\,D\,t}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-\xi)^2}{4\,D\,t}\Big)\,h(\xi)\,d\xi\,. \tag{II.1.15}$$

II.2 Exponential Fourier transform for the inhomogeneous wave equation

II.3 Sine and cosine Fourier transforms

Fourier transforms in sine and cosine are well adapted tools to solve PDEs over semi-infinite bodies, e.g.

$$0 \; \boxed{\qquad\qquad\qquad} \cdots \quad \infty \tag{II.3.1}$$

They stem from the integral Fourier theorem,

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\int_{-\infty}^{\infty} e^{-i\alpha\xi} f(\xi)\,d\xi\; d\alpha \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(\cos(\alpha(x-\xi)) + i\,\overbrace{\sin(\alpha(x-\xi))}^{\text{odd in }\alpha}\Big) f(\xi)\,d\xi\; d\alpha \\
&= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}\cos(\alpha(x-\xi))\, f(\xi)\,d\xi\; d\alpha \\
&= \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}\Big(\cos(\alpha x)\,\cos(\alpha\xi) + \sin(\alpha x)\,\sin(\alpha\xi)\Big) f(\xi)\,d\xi\; d\alpha \\
&= \frac{1}{\pi}\int_{0}^{\infty}\cos(\alpha x)\int_{-\infty}^{\infty}\cos(\alpha\xi)\; f(\xi)\,d\xi\; d\alpha \\
&+ \frac{1}{\pi}\int_{0}^{\infty}\sin(\alpha x)\int_{-\infty}^{\infty}\sin(\alpha\xi)\; f(\xi)\,d\xi\; d\alpha\,.
\end{aligned} \tag{II.3.2}$$

II.3.1 Sine Fourier transforms for odd $f(x)$ in $]-\infty,\infty[$

If the function $f(x)$ is odd over $]-\infty,\infty[$, or, if it is initially defined over $[0,\infty[$, and extended to an odd function over $]-\infty,\infty[$, then (II.3.2) yields,

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\sin(\alpha x)\int_{0}^{\infty}\sin(\alpha\xi)\; f(\xi)\,d\xi\; d\alpha\,, \tag{II.3.3}$$

expression which motivates the definition of a sine transform, and by the same token, of its inverse,

$$F_S(\alpha) = \int_{0}^{\infty}\sin(\alpha x)\, f(x)\,dx, \quad f(x) = \frac{2}{\pi}\int_{0}^{\infty}\sin(\alpha x)\;\; F_S(\alpha)\,d\alpha\,, \tag{II.3.4}$$

II.3.2 Cosine Fourier transforms for even $f(x)$ in $]-\infty,\infty[$

If the function $f(x)$ is even over $]-\infty,\infty[$, or, if it is initially defined over $[0,\infty[$, and extended to an even function over $]-\infty,\infty[$, then (II.3.2) yields,

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\cos(\alpha x)\int_{0}^{\infty}\cos(\alpha\xi)\; f(\xi)\,d\xi\; d\alpha\,, \tag{II.3.5}$$

expression which motivates the definition of a cosine transform, and by the same token, of its inverse,

$$F_C(\alpha) = \int_{0}^{\infty}\cos(\alpha x)\, f(x)\,dx, \quad f(x) = \frac{2}{\pi}\int_{0}^{\infty}\cos(\alpha x)\;\; F_C(\alpha)\,d\alpha\,. \tag{II.3.6}$$

Which of these two transforms is more appropriate to solve PDEs over semi-infinite bodies? The answer depends on the boundary conditions, as will be seen now.

II.3.3 Rules for derivatives

The following transforms of a derivative are easily established by simple integration by parts, accounting that $f(x)$ tends to 0 at $\pm\infty$,

$$\begin{aligned}
\mathcal{F}_C\Big\{\frac{\partial f(x)}{\partial x}\Big\}(\alpha) &= \alpha\,\mathcal{F}_S\{f(x)\}(\alpha) - f(0),\\
\mathcal{F}_S\Big\{\frac{\partial f(x)}{\partial x}\Big\}(\alpha) &= -\alpha\,\mathcal{F}_C\{f(x)\}(\alpha),\\
\mathcal{F}_C\Big\{\frac{\partial^2 f(x)}{\partial x^2}\Big\}(\alpha) &= -\alpha^2\,\mathcal{F}_C\{f(x)\}(\alpha) - \frac{\partial f}{\partial x}(0),\\
\mathcal{F}_S\Big\{\frac{\partial^2 f(x)}{\partial x^2}\Big\}(\alpha) &= -\alpha^2\,\mathcal{F}_S\{f(x)\}(\alpha) + \alpha\, f(0)\,.
\end{aligned} \tag{II.3.7}$$

Therefore, since diffusion equations involve a second order derivative wrt space,
- a boundary condition at $x = 0$ in $\partial f(0)/\partial x$ is accounted for by the cosine transform;
- a boundary condition at $x = 0$ in $f(0)$ is accounted for by the sine transform.

II.4 Sine and cosine Fourier transforms to solve IBVPs in semi-infinite bodies

We consider a diffusion phenomenon in a semi-infinite bar, aligned with the x-axis, and endowed with a diffusion coefficient $D > 0$. For definiteness, the unknown field may be interpreted as a temperature. The initial temperature is a known function of space. The radiation condition imposes the temperature to vanish at infinity. The governing equations

$$\begin{aligned}
&\text{field equation (FE)} \quad \frac{\partial u}{\partial t} - D\,\frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0,\ x > 0;\\
&\text{initial condition (IC)} \quad u(x,0) = h(x), \quad 0 \leq x < \infty\,;\\
&\text{radiation condition (RC)} \quad u(x \to \infty, t) = 0\,,
\end{aligned} \tag{II.4.1}$$

will be completed by two types of boundary conditions.

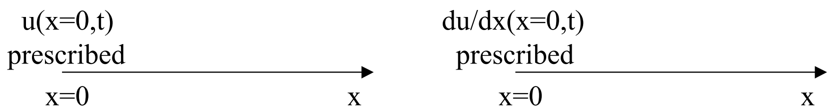

Figure II.2 The boundary conditions considered consist in prescribing either the primary unknown or its space derivative, interpreted as a flux.

II.4.1 Prescribed flux at $x = 0$

We first consider a situation where the flux is prescribed,

$$\text{boundary condition (BC)} \quad \frac{\partial u}{\partial x}(x = 0, t) = f(t), \quad t > 0\,. \tag{II.4.2}$$

The solution,

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi\,D\,t}}\int_{-\infty}^{\infty}\exp\Big(-\frac{(x-y)^2}{4\,D\,t}\Big)\,h(y)\,dy \\ &- \sqrt{\frac{D}{\pi}}\int_0^t \exp\Big(\frac{-x^2}{4\,D\,(t-\tau)}\Big)\,\frac{f(\tau)}{\sqrt{t-\tau}}\,d\tau\,, \end{aligned} \tag{II.4.3}$$

where the function $h(y>0)$ has been extended to $y<0$ as an **even** function, clearly evidences the contributions of the initial condition and boundary condition. Incidentally, note that these contributions simply sum, since the problem is linear.

Proof:

The proof is a bit lengthy and tedious, but otherwise straightforward. Since the boundary condition prescribes a flux, the problem is solved through the cosine Fourier transform,

$$u(x,t)\to U_c(\alpha,t)=\mathcal{F}_C\{u(x,t)\}(\alpha,t)\,. \tag{II.4.4}$$

Therefore,

$$\begin{aligned} &\text{(FE)}\ \frac{\partial U_C}{\partial t}(\alpha,t)+D\,\alpha^2\,U_C(\alpha,t)=-D\,\frac{\partial u(0,t)}{\partial x}\overset{\text{(BC)}}{=}-D\,f(t)\,,\quad t>0, \\ &\text{(CI)}\ U_C(\alpha,t)=H_C(\alpha)\,, \end{aligned} \tag{II.4.5}$$

which, switching to total derivative in time with the Fourier variable being seen as a parameter, is easily integrated to

$$\begin{aligned} &e^{-\alpha^2\,D\,t}\,\frac{d}{dt}(e^{\alpha^2\,D\,t}\,U_C(\alpha,t))=-D\,f(t)\,,\quad t>0, \\ &U_C(\alpha,t)=H_C(\alpha)\,e^{-\alpha^2\,D\,t}-D\int_0^t e^{-D\alpha^2\,(t-\tau)}\,f(\tau)\,d\tau\,, \end{aligned} \tag{II.4.6}$$

The inverse is

$$u(x,t)=\frac{2}{\pi}\int_0^{\infty}\cos(\alpha\,x)\,H_C(\alpha)\,e^{-\alpha^2\,D\,t}\,d\alpha-\frac{2}{\pi}\,D\int_0^{\infty}\cos(\alpha\,x)\int_0^t e^{-D\alpha^2\,(t-\tau)}\,f(\tau)\,d\tau\,d\alpha\,. \tag{II.4.7}$$

The first term can be manipulated as follows,

$$\begin{aligned} &\frac{2}{\pi}\int_0^{\infty}\cos(\alpha\,x)\,H_C(\alpha)\,e^{-\alpha^2\,D\,t}\,d\alpha \\ &=\frac{2}{\pi}\int_0^{\infty}\cos(\alpha\,x)\,e^{-\alpha^2\,D\,t}\int_0^{\infty}\cos(\alpha\,y)\,h(y)\,dy\,d\alpha \\ &=\int_0^{\infty}h(y)\,\frac{1}{\pi}\int_0^{\infty}e^{-\alpha^2\,D\,t}\Big(\cos\,(\alpha\,(y-x))+\cos\,(\alpha\,(y+x))\Big)\,d\alpha\,dy \\ &=\frac{1}{2\sqrt{\pi\,D\,t}}\int_0^{\infty}h(y)\,\Big(\exp\Big(-\frac{(y-x)^2}{4\,D\,t}\Big)+\exp\Big(-\frac{(y+x)^2}{4\,D\,t}\Big)\Big)\,dy\quad\text{using }(II.1.9) \\ &=\frac{1}{2\sqrt{\pi\,D\,t}}\Big(\int_0^{\infty}h(y)\,\exp\Big(-\frac{(y-x)^2}{4\,D\,t}\Big)\,dy+\overbrace{\int_{-\infty}^{0}h(-y)\,\exp\Big(-\frac{(y-x)^2}{4\,D\,t}\Big)\,dy}^{y\to-y}\Big), \\ &=\frac{1}{2\sqrt{\pi\,D\,t}}\int_{-\infty}^{\infty}h(y)\,\exp\Big(-\frac{(y-x)^2}{4\,D\,t}\Big)\,dy\,, \end{aligned} \tag{II.4.8}$$

where the function $h(y>0)$ has been extended to $y<0$ as an even function.

The second term can be transformed as well,

$$
\begin{aligned}
&-\frac{2}{\pi} D \int_0^\infty \cos(\alpha\, x) \int_0^t e^{-D\alpha^2\,(t-\tau)}\, f(\tau)\, d\tau\, d\alpha \\
&= -\sqrt{\frac{D}{\pi}} \int_0^t \frac{f(\tau)}{\sqrt{t-\tau}} \exp\Big(\frac{-x^2}{4\,D\,(t-\tau)}\Big)\, d\tau\,,
\end{aligned}
\tag{II.4.9}
$$

using (II.1.9). □

II.4.2 Prescribed temperature at $x = 0$

We consider now a situation where the temperature is prescribed,

$$
\text{boundary condition (BC)} \quad u(x=0,t) = g(t), \quad t>0\,. \tag{II.4.10}
$$

The solution,

$$
\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi\, D\, t}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-y)^2}{4\,D\,t}\Big)\, h(y)\, dy \\
&+ \frac{x}{2\sqrt{D\,\pi}} \int_0^t \exp\Big(-\frac{x^2}{4\,D\,(t-\tau)}\Big) \frac{g(\tau)}{(t-\tau)^{3/2}}\, d\tau\,,
\end{aligned}
\tag{II.4.11}
$$

may also be written in a format that highlights a relation with the previous problem,

$$
\begin{aligned}
u(x,t) &= \frac{1}{2\sqrt{\pi\, D\, t}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-y)^2}{4\,D\,t}\Big)\, h(y)\, dy \\
&+ \Big(\frac{\partial}{\partial x}\Big)\Big(-\sqrt{\frac{D}{\pi}} \int_0^t \exp\Big(\frac{-x^2}{4\,D\,(t-\tau)}\Big) \frac{g(\tau)}{\sqrt{t-\tau}}\, d\tau\Big)\,,
\end{aligned}
\tag{II.4.12}
$$

where the function $h(y>0)$ has been extended to $y<0$ as an **odd** function.

Proof:

Since the boundary condition prescribes the primary unknown, the problem is solved through the sine Fourier transform,

$$
u(x,t) \to U_S(\alpha,t) = \mathcal{F}_S\{u(x,t)\}(\alpha,t)\,. \tag{II.4.13}
$$

Therefore,

$$
\begin{aligned}
&\text{(FE)}\ \frac{\partial U_S}{\partial t}(\alpha,t) + D\,\alpha^2\, U_S(\alpha,t) = D\,\alpha\, u(0,t) \overset{\text{(BC)}}{=} D\,\alpha\, g(t)\,, \quad t>0, \\
&\text{(CI)}\ U_S(\alpha,t) = H_S(\alpha)\,,
\end{aligned}
\tag{II.4.14}
$$

which, switching to total derivative in time with the Fourier variable being seen as a parameter, is easily integrated to

$$
U_S(\alpha,t) = H_S(\alpha)\, e^{-\alpha^2\, D\, t} + D\,\alpha \int_0^t e^{-D\alpha^2\,(t-\tau)}\, g(\tau)\, d\tau\,, \quad t>0\,. \tag{II.4.15}
$$

The inverse is

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \sin(\alpha\, x)\, H_S(\alpha)\, e^{-\alpha^2 D\, t}\, d\alpha + \frac{2}{\pi}\, D \int_0^\infty \sin(\alpha\, x) \int_0^t e^{-D\alpha^2 (t-\tau)}\, g(\tau)\, d\tau\, \alpha\, d\alpha\,. \tag{II.4.16}$$

The first term can be manipulated as follows,

$$\begin{aligned}
&\frac{2}{\pi} \int_0^\infty \sin(\alpha\, x)\, H_S(\alpha)\, e^{-\alpha^2 D\, t}\, d\alpha \\
&= \int_0^\infty h(y)\, \frac{1}{\pi} \int_0^\infty e^{-\alpha^2 D\, t} \Big(- \cos\,(\alpha\,(y+x)) + \cos\,(\alpha\,(y-x)) \Big)\, d\alpha\, dy \\
&= \frac{1}{2\sqrt{\pi\, D\, t}} \Big(\int_0^\infty h(y)\, \exp\Big(- \frac{(y-x)^2}{4\, D\, t} \Big)\, dy - \int_{-\infty}^0 h(-y)\, \exp\Big(- \frac{(y-x)^2}{4\, D\, t} \Big)\, dy \Big) \\
&= \frac{1}{2\sqrt{\pi\, D\, t}} \int_{-\infty}^\infty h(y)\, \exp\Big(- \frac{(y-x)^2}{4\, D\, t} \Big)\, dy\,,
\end{aligned} \tag{II.4.17}$$

where the function $h(y>0)$ has been extended to $y<0$ as an odd function. Use has been made of (II.1.9) to go from the 2nd line to the 3rd line.

The second term can be transformed as well,

$$\begin{aligned}
&\frac{2}{\pi}\, D \int_0^\infty \sin(\alpha\, x) \int_0^t e^{-D\alpha^2 (t-\tau)}\, g(\tau)\, d\tau\, \alpha\, d\alpha \\
&= \frac{2}{\pi}\, D \int_0^t g(\tau) \Big(- \frac{\partial}{\partial x} \Big) \Big(\int_0^\infty \cos(\alpha\, x)\, e^{-D\alpha^2 (t-\tau)}\, d\alpha \Big)\, d\tau \\
&= \Big(\frac{\partial}{\partial x} \Big) \Big(- \sqrt{\frac{D}{\pi}} \int_0^t \frac{g(\tau)}{\sqrt{t-\tau}}\, \exp\Big(\frac{-x^2}{4\, D\, (t-\tau)} \Big)\, d\tau \Big) \\
&= \frac{x}{2\sqrt{\pi\, D}} \int_0^t \frac{g(\tau)}{(t-\tau)^{3/2}}\, \exp\Big(- \frac{x^2}{4\, D\, (t-\tau)} \Big)\, d\tau\,.
\end{aligned} \tag{II.4.18}$$

Use has been made of (II.1.9) to go from the 2nd line to the 3rd line. The latter line indicates that the second term in this section could have been guessed by applying the operator $\partial/\partial x$ to the result of the previous section. □

Particular case:

$$\begin{aligned}
&\text{initial condition (IC)} \quad u(x,0) = h(x) = 0, \quad 0 \le x < \infty\,; \\
&\text{boundary condition (BC)} \quad u(x=0,t) = g(t) = u_0, \quad t > 0\,.
\end{aligned} \tag{II.4.19}$$

With the change of variable,

$$\tau \;\rightarrow \eta = \frac{x}{2\sqrt{D\,(t-\tau)}} \qquad \Rightarrow \qquad \frac{\partial \eta}{\partial \tau} = \frac{x}{2\sqrt{D}\,(t-\tau)^{3/2}}\,, \tag{II.4.20}$$

the solution (II.4.11) becomes

$$u(x,t) = u_0\, \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{D\, t}}^\infty e^{-\eta^2}\, d\eta = u_0\, \mathrm{erfc}\Big(\frac{x}{2\sqrt{D\, t}} \Big)\,. \tag{II.4.21}$$

Note the derivative

$$\frac{\partial u(x,t)}{\partial x} = \frac{-u_0}{\sqrt{\pi\, D\, t}}\, \exp\Big(- \frac{x^2}{4\, D\, t} \Big)\,. \tag{II.4.22}$$

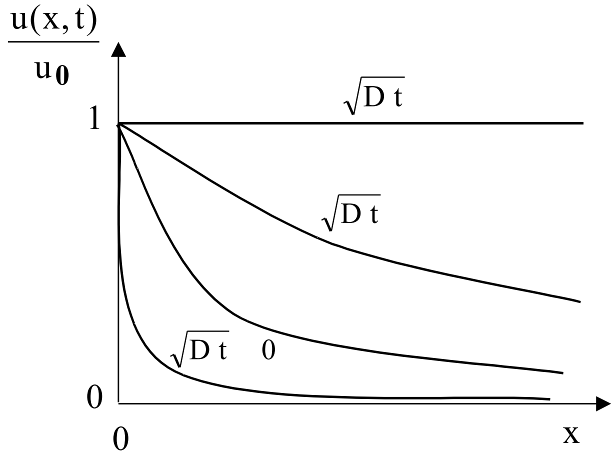

Figure II.3 A semi-infinite bar is subject to a heat shock at its boundary $x = 0$ at time $t = 0$. Spatial profile of the temperature at various times.

II.5 Two general algebraic relations with physical relevance

The following algebraic relations have far-reaching consequences in mathematical physics. Still, we will be content to show a mere academic application [2].

II.5.1 Parseval identities

Parseval identities indicate that the scalar products in the original and transformed spaces are conserved.

Under suitable conditions for the real functions $f(x)$ and $g(x)$, and with the standard notation for their complex transforms $F(\alpha)$ and $G(\alpha)$, cosine transforms $F_C(\alpha)$ and $G_C(\alpha)$, and sine transforms $F_S(\alpha)$ and $G_S(\alpha)$, respectively, Parseval identities can be cast in the formats,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\, g(x)\, dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha)\, \overline{G}(\alpha)\, d\alpha \\
\int_{0}^{\infty} f(x)\, g(x)\, dx &= \frac{2}{\pi} \int_{0}^{\infty} F_C(\alpha)\, G_C(\alpha)\, d\alpha \\
\int_{0}^{\infty} f(x)\, g(x)\, dx &= \frac{2}{\pi} \int_{0}^{\infty} F_S(\alpha)\, G_S(\alpha)\, d\alpha
\end{aligned} \tag{II.5.1}$$

When $f = g$, Parseval relations can be interpreted as indicating that

energy is invariant under Fourier transforms

We may offer a simple algebraic consequence of these relations, to calculate a 'difficult' integral from a simple one. For example, with the preliminary transform, with $a > 0$,

$$f(x) = e^{-a,x}\, \mathcal{H}(x) \to F(\alpha) = \frac{1}{a + i\,\alpha}\,, \tag{II.5.2}$$

[2]This section should be part of the basics of Fourier analysis. It is here because it uses the three Fourier transforms at once. Please skip the section for now.

results

$$\int_{-\infty}^{\infty} \frac{1}{|a+i\,\alpha|^2}\,d\alpha = 2\,\pi \int_0^{\infty} e^{-2ax}\,d\alpha = \frac{\pi}{a}\,, \tag{II.5.3}$$

which can be easily checked,

$$\int_{-\infty}^{\infty} \frac{1}{a^2+\alpha^2}\,d\alpha = \frac{1}{a}\left[\tan^{-1}\left(\frac{\alpha}{a}\right)\right]_{-\infty}^{\infty} = \frac{\pi}{a}\;! \tag{II.5.4}$$

Similarly, using the real Fourier transforms of the very same function, Exercise II.4,

$$\mathcal{F}_C\{e^{-a\,x}\}(\alpha) = \frac{a}{a^2+\alpha^2}\,, \quad \mathcal{F}_S\{e^{-a\,x}\}(\alpha) = \frac{\alpha}{a^2+\alpha^2}\,, \tag{II.5.5}$$

we can estimate two further 'difficult' integrals,

$$\begin{aligned}\int_0^{\infty} \left(\frac{1}{a^2+\alpha^2}\right)^2 d\alpha &= \frac{\pi}{2a^2}\int_0^{\infty}\left(e^{-ax}\right)^2 d\alpha = \frac{\pi}{4\,a^3}\\ \int_0^{\infty} \left(\frac{\alpha}{a^2+\alpha^2}\right)^2 d\alpha &= \frac{\pi}{2}\int_0^{\infty}\left(e^{-ax}\right)^2 d\alpha = \frac{\pi}{4\,a}\,.\end{aligned} \tag{II.5.6}$$

Note that, as they should, the two relations (II.5.6) associated with the cosine (real part) and sine (imaginary part) Fourier transforms, imply (II.5.4), associated with the complex Fourier transform.

II.5.2 Heisenberg uncertainty principle

The basic idea here is that

the smaller the support of a function,
the larger the support of its Fourier transforms,
and conversely

Perhaps, the most conspicuous example is the Dirac distribution. This property is coined 'Heisenberg uncertainty principle' as an analogy to the fact that one can not estimate with the same accuracy position and momentum of a particle. Improving on one side implies worsening on the other side.

For a function $f(x)$ with Fourier transform $F(\alpha)$, the relation is given the following algebraic expression,

$$W_x\,W_\alpha \geq 1\,, \tag{II.5.7}$$

with

$$W_x = 2\,\frac{\displaystyle\int_{-\infty}^{\infty} x^2\,|f(x)|^2\,dx}{\displaystyle\int_{-\infty}^{\infty} |f(x)|^2\,dx}, \quad W_\alpha = 2\,\frac{\displaystyle\int_{-\infty}^{\infty} \alpha^2\,|F(\alpha)|^2\,d\alpha}{\displaystyle\int_{-\infty}^{\infty} |F(\alpha)|^2\,d\alpha}\,. \tag{II.5.8}$$

The proof goes as follows. It starts from Cauchy-Schwarz inequality,

$$X^2 \equiv |\int_{-\infty}^{\infty} x\,f(x)\,f'(x)\,dx|^2 \leq \int_{-\infty}^{\infty} |x\,f(x)|^2\,dx \int_{-\infty}^{\infty} |f'(x)|^2\,dx\,. \tag{II.5.9}$$

Now, by integration by part, assuming $f(x)$ to be decrease sufficiently at infinity, and using Parseval relation,

$$\begin{aligned}
X = \int_{-\infty}^{\infty} x\, f(x)\, f'(x)\, dx \quad &= \quad -\frac{1}{2}\int_{-\infty}^{\infty} |f(x)|^2\, dx + \frac{1}{2}\,\overbrace{\Big[x\, f(x)^2\Big]_{-\infty}^{\infty}}^{=0} \\
&= \quad \overbrace{-\frac{1}{2}\int_{-\infty}^{\infty} |f(x)|^2\, dx}^{A} \\
&\overset{\text{Parseval}}{=} \quad \overbrace{-\frac{1}{4\,\pi}\int_{-\infty}^{\infty} |F(\alpha)|^2\, d\alpha}^{B}\,.
\end{aligned} \tag{II.5.10}$$

Using again Parseval relation, and the rule of the transform of a derivative,

$$\begin{aligned}
\int_{-\infty}^{\infty} |f'(x)|^2\, dx \quad &\overset{\text{Parseval}}{=} \quad \frac{1}{2\,\pi}\int_{-\infty}^{\infty} |\mathcal{F}\{f'(x)\}(\alpha)|^2\, d\alpha \\
&\overset{f'(x)\to i\alpha\, F(\alpha)}{=} \quad \overbrace{\frac{1}{2\,\pi}\int_{-\infty}^{\infty} |\alpha\, F(\alpha)|^2\, d\alpha}^{C}\,.
\end{aligned} \tag{II.5.11}$$

The inequality is finally deduced by inserting the previous relations into (II.5.9),

$$\begin{aligned}
X^2 \quad &= \quad \overbrace{\Big(-\frac{1}{2}\int_{-\infty}^{\infty} |f(x)|^2\, dx\Big)}^{A}\ \overbrace{\Big(-\frac{1}{4\,\pi}\int_{-\infty}^{\infty} |F(\alpha)|^2\, d\alpha\Big)}^{B} \\
&\overset{\text{(II.5.9)}}{\leq} \quad \int_{-\infty}^{\infty} |x\, f(x)|^2\, dx\ \overbrace{\frac{1}{2\,\pi}\int_{-\infty}^{\infty} |\alpha\, F(\alpha)|^2\, d\alpha}^{C}\,.
\end{aligned} \tag{II.5.12}$$

□

II.6 Some basic information on plane strain elasticity

This section serves as a brief introduction to Exercice 5, that addresses a problem of plane strain elasticity in the half-plane.

Let me list first the basic equations of plane strain elasticity, before introducing the Airy stress function.

Static equilibrium with vanishing body forces

Static equilibrium expresses in terms of the Cauchy stress $\boldsymbol{\sigma}$ with components σ_{xx}, σ_{yy}, and $\sigma_{xy} = \sigma_{yx}$, in the cartesian axes (x, y). Cauchy stress satisfies, at each point inside the body Ω, the field equation $\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$, namely componentwise,

$$\left.\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0
\end{aligned}\right\}\quad \text{in } \Omega \tag{II.6.1}$$

Plane strain in the plane (x, y)

The infinitesimal strain $\boldsymbol{\epsilon}$ with components ϵ_{xx}, ϵ_{yy}, and $\epsilon_{xy} = \epsilon_{yx}$, in the cartesian axes (x, y), expresses in terms of the displacement field $\mathbf{u} = (u_x, u_y)$, namely $\boldsymbol{\epsilon} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$, or componentwise,

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \epsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad 2\,\epsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}. \tag{II.6.2}$$

Compatibility condition

Since there are only two displacement components, and three strain components, the latter obey a relation,

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\,\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}. \tag{II.6.3}$$

Plane strain elasticity

For a compressible isotropic elastic body, the strain and stress tensors are linked by a one-to-one relation, phrased in terms of the Young's modulus $E > 0$, and Poisson's ratio $\nu \in]-1, 1/2[$,

$$\begin{aligned} E\,\epsilon_{xx} &= \sigma_{xx} - \nu\,(\sigma_{yy} + \sigma_{zz}) \\ E\,\epsilon_{yy} &= \sigma_{yy} - \nu\,(\sigma_{xx} + \sigma_{zz}) \\ 2\,E\,\epsilon_{xy} &= 2\,(1+\nu)\,\sigma_{xy}\,. \end{aligned} \tag{II.6.4}$$

Since the out-of-plane strain component ϵ_{zz} vanishes,

$$E\,\epsilon_{zz} = \sigma_{zz} - \nu\,(\sigma_{xx} + \sigma_{yy}) = 0\,, \tag{II.6.5}$$

the out-of-plane stress component σ_{zz} does *not* vanish. Substituting the resulting value in the constitutive equations (II.6.4), and inserting the strain components in the compatibility relation (II.6.3) yields the field equation,

$$\Delta(\sigma_{xx} + \sigma_{yy}) = 0\,. \tag{II.6.6}$$

Airy stress function

The static equilibrium is automatically satisfied if the stress components are expressed in terms of the Airy stress function $\phi(x, y)$,

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}. \tag{II.6.7}$$

The sole condition to be satisfied is the compatibility condition (II.6.6), which in fact becomes a **biharmonic equation** for the Airy stress function,

$$\Delta\Delta\phi(x, y) = 0, \quad (x, y) \in \Omega\,. \tag{II.6.8}$$

Exercise II.1: **drainage of an infinite porous medium, an elliptic PDE**.

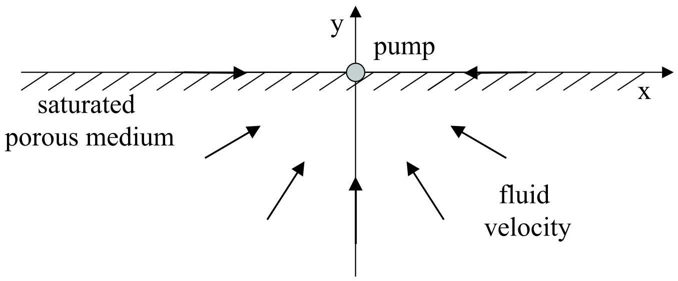

Figure II.4 A half-space is drained by a linear pump located on the free surface.

The lower half-space $y < 0$ is constituted by a porous medium, which is saturated by water. Seepage is induced by a rectilinear drain, aligned with the axis z, that pumps water at a given constant flow rate Q.

The issue is to derive the velocity $\mathbf{v}$ of the fluid in the lower half-space. Since the drain is infinite in the z-direction, the velocity does not depend on the out-of-plane coordinate z.

Water is assumed to be incompressible, so that its velocity $\mathbf{v}$ is divergence free,

$$\operatorname{div} \mathbf{v} = 0\,. \tag{1}$$

Seepage is governed by Darcy law that relates the fluid velocity to the gradient of a scalar potential ϕ,

$$\mathbf{v} = \nabla\phi\,, \tag{2}$$

which is contributed by the fluid pressure and the potential energy. The coordinates have been scaled so as to absorb the hydraulic conductivity.

The equations governing the seepage problem are,

$$\begin{aligned} &\text{(FE) field equation} \quad \operatorname{div}\nabla\phi = \Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad x \in]-\infty,\infty[,\ y < 0; \\ &\text{(BC)}_1 \text{ boundary condition} \quad \mathbf{v} = \nabla\phi \to 0, \quad \text{as} \quad x^2 + y^2 \to \infty\,; \\ &\text{(BC)}_2 \text{ boundary condition} \quad v_y = \frac{\partial\phi}{\partial y}(x,0) = Q\,\delta(x)\,. \end{aligned} \tag{3}$$

Derive the potential $\phi(x, y)$ and show that the fluid velocity is purely radial. Check that the solution satisfies the boundary conditions.

Solution:

Since the space variable x varies between $-\infty$ and $+\infty$, the problem will be solved via the exponential Fourier transform $x \to \alpha$,

$$\phi(x,y) \to \Phi(\alpha,y) = \mathcal{F}\{\phi(x,y)\}(\alpha)\,, \tag{4}$$

and we admit that the operators Fourier transform and partial derivative wrt y commute,

$$\frac{\partial}{\partial y}\mathcal{F}\{\phi(x,y)\}(\alpha) = \mathcal{F}\{\frac{\partial}{\partial y}\phi(x,y)\}(\alpha)\,. \tag{5}$$

The transforms of the field equation and boundary conditions,

$$\begin{aligned} &\text{(FE)} \ \frac{\partial^2 \Phi}{\partial y^2}(\alpha, y) - |\alpha|^2\, \Phi(\alpha, y) = 0, \quad y < 0, \\ &\text{(BC)}_1 \ \frac{\partial \Phi}{\partial y}(\alpha, y \to \infty) = 0, \\ &\text{(BC)}_2 \ \frac{\partial \Phi}{\partial y}(\alpha, y = 0) = Q\,, \end{aligned} \tag{6}$$

solve to

$$\Phi(\alpha, y) = Q\, \frac{e^{|\alpha|\, y}}{|\alpha|}\,. \tag{7}$$

Note the trick used to introduce the absolute value in $(6)_1$.

The inverse Fourier transform can be easily manipulated,

$$\begin{aligned} \phi(x, y) &= \frac{Q}{2\,\pi} \int_{-\infty}^{+\infty} \frac{e^{i\alpha\, x + |\alpha|\, y}}{|\alpha|}\, d\alpha \\ &= \frac{Q}{2\,\pi} \Big(\int_{-\infty}^{0} \frac{e^{i\alpha\, x - \alpha\, y}}{(-\alpha)}\, d\alpha + \int_{0}^{+\infty} \frac{e^{i\alpha\, x + \alpha\, y}}{\alpha}\, d\alpha \Big) \\ &= \frac{Q}{\pi} \int_{0}^{+\infty} \frac{e^{\alpha\, y}}{\alpha} \cos(\alpha\, x)\; d\alpha\,. \end{aligned} \tag{8}$$

The latter integral can be estimated by first taking the derivative wrt y,

$$\begin{aligned} \frac{\partial \phi}{\partial y}(x, y) &= \frac{Q}{\pi} \int_{0}^{+\infty} e^{\alpha\, y} \cos(\alpha\, x)\; d\alpha \\ &= \frac{Q}{2\,\pi} \int_{0}^{+\infty} (e^{i\,\alpha\, x + \alpha\, y} + e^{-i\,\alpha\, x + \alpha\, y})\, d\alpha \\ &= -\frac{Q}{2\,\pi} \Big(\frac{1}{i\, x + y} + \frac{1}{-i\, x + y} \Big) \\ &= -\frac{Q}{\pi}\, \frac{y}{r^2}, \end{aligned} \tag{9}$$

with $r^2 = x^2 + y^2$. Integration wrt y yields finally

$$\phi(x, y) = -\frac{Q}{\pi} \,\mathrm{Ln}\, r, \tag{10}$$

to within a constant, and

$$v_r = \frac{\partial \phi}{\partial r} = -\frac{Q}{\pi}\, \frac{1}{r}\,. \tag{11}$$

Note that, for pumping $Q > 0$, the velocity vector points to the pump as expected. Moreover, it has been shown in the Chapter devoted to distributions that, in the sense of distributions,

$$\lim_{\epsilon \to 0^+} \frac{1}{\pi}\, \frac{\epsilon}{x^2 + \epsilon^2} = \delta(x)\,, \tag{12}$$

Therefore, setting here $\epsilon = -y > 0$, the solution (9) is seen to satisfy the boundary condition $(\mathrm{BC})_2$.

Exercise II.2: **potential flow in the upper half-plane**.

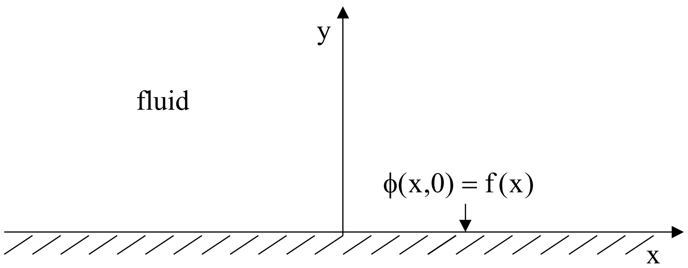

Figure II.5 The flow potential in the upper half-plane is prescribed along the x-axis.

A fluid fills the upper half-plane $y > 0$. The equations governing the flow potential of the fluid are,

$$\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad x \in]-\infty, \infty[, \ y > 0; \\
&\text{(BC) boundary condition} \quad \phi(x, y = 0) = f(x), \quad x \in]-\infty, \infty[\,; \\
&\text{(RC) radiation condition} \quad \phi(x, y) \text{ bounded}, \quad x^2 + y^2 \to \infty\,.
\end{aligned} \tag{1}$$

Derive the flow potential, sometimes referred to as a Poisson's formula for the half-plane,

$$\phi(x, y) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{f(\xi)}{y^2 + (x - \xi)^2}\, d\xi\,. \tag{2}$$

A sort of inverse problem is proposed in Exercise II.4: given the potential $\phi(x, y)$, the issue is to identify the boundary data $f(x)$.

Solution:

Since the space variable x varies between $-\infty$ and $+\infty$, the problem will be solved via the exponential Fourier transform $x \to \alpha$, and the operators Fourier transform and partial derivative wrt y are assumed to commute.

The transforms of the field equation and boundary conditions take the form,

$$\begin{aligned}
&\text{(FE)} \ \frac{\partial^2 \Phi}{\partial y^2}(\alpha, y) - |\alpha|^2\, \Phi(\alpha, y) = 0, \quad y > 0, \\
&\text{(BC)} \ \Phi(\alpha, y = 0) = F(\alpha) \\
&\text{(RC)} \ \Phi(\alpha, y) \text{ bounded } \forall y \geq 0\,.
\end{aligned} \tag{3}$$

Instead of using directly the radiation condition, the Fourier transform itself is required to remain bounded at infinity.

The problem solves to

$$\Phi(\alpha, y) = e^{-|\alpha|\, y}\, F(\alpha)\,. \tag{4}$$

Note the trick used to introduce the absolute value in $(3)_1$.

The inverse Fourier transform can be easily manipulated,

$$
\begin{aligned}
\phi(x,y) &= \frac{1}{2\,\pi}\int_{-\infty}^{+\infty} e^{i\alpha\,x-|\alpha|\,y}\int_{-\infty}^{+\infty} e^{-i\alpha\,\xi}\,f(\xi)\,d\xi\;d\alpha \\
&= \frac{1}{2\,\pi}\int_{-\infty}^{+\infty} f(\xi)\Big(\int_{-\infty}^{0} e^{(y+i,(x-\xi))\,\alpha}\;d\alpha+\int_{0}^{+\infty} e^{(-y+i,(x-\xi))\,\alpha}\;d\alpha\Big)\;d\xi \\
&= \frac{1}{2\,\pi}\int_{-\infty}^{+\infty} f(\xi)\Big(\frac{1}{y+i\,(x-\xi)}+\frac{1}{y-i\,(x-\xi)}\Big)\;d\xi \\
&= \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f(\xi)}{y^2+(x-\xi)^2}\;d\xi\,.
\end{aligned}
\tag{5}
$$

Exercise II.3: **potential flow in a semi-infinite strip of finite width**.

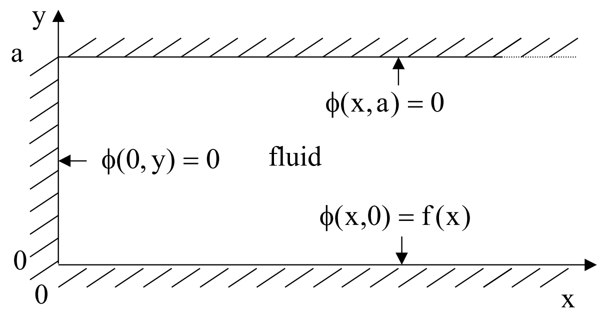

Figure II.6 Flow potential in a semi-infinite strip of finite width with prescribed data along the boundaries.

A fluid fills a semi-infinite strip Ω of finite width a,

$$\Omega = \{x \geq 0; \quad 0 \leq y \leq a\}\,. \tag{1}$$

The equations governing the flow potential $\phi(x, y)$ of the fluid are,

$$\begin{aligned}
&\text{(FE) field equation} && \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad x > 0; \quad 0 < y < a; \\
&\text{(BC) boundary conditions} && \phi(x, y = 0) = f(x), \quad x \in [0, \infty[\\
& && \phi(x, y = a) = 0, \quad x \in [0, \infty[\\
& && \phi(x = 0, y) = 0, \quad y \in [0, a]\,; \\
&\text{(RC) radiation condition} && \phi(x, y) \text{ bounded}, \quad \forall (x, y) \in \Omega\,.
\end{aligned} \tag{2}$$

Show that, when the width a is very large, the flow potential can be cast in the format,

$$\phi(x, y) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\text{sgn}(\xi) f(|\xi|)}{y^2 + (x - \xi)^2}\, d\xi\,. \tag{3}$$

Solution:

Since the space variable x varies between 0 and $+\infty$, and the unknown is prescribed on the boundary, the problem will be solved via the sine Fourier transform $x \to \alpha$, and the operators Fourier transform and partial derivative wrt y are assumed to commute.

The transforms of the field equation and boundary conditions take the form,

$$\begin{aligned}
&\text{(FE)} \quad \frac{\partial^2 \Phi_S}{\partial y^2}(\alpha, y) - \alpha^2\, \Phi_S(\alpha, y) + \alpha\, \overbrace{\phi(0, y)}^{=0,\text{(BC)}} = 0, \quad 0 < y < a, \\
&\text{(BC)} \quad \Phi_S(\alpha, y = 0) = F_S(\alpha) \\
&\qquad\quad\; \Phi_S(\alpha, y = a) = 0\,.
\end{aligned} \tag{4}$$

The problem solves to (charitable reminder $\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b$, $\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b$),

$$\Phi_S(\alpha, y) = F_S(\alpha)\,\frac{\sinh(\alpha(a-y))}{\sinh(\alpha\, a)}\,, \tag{5}$$

with inverse

$$\phi(x,y) = \frac{2}{\pi}\int_0^\infty \sin(\alpha\, x)\int_0^\infty \sin(\alpha\,\xi)\, f(\xi)\, d\xi\,\frac{\sinh(\alpha(a-y))}{\sinh(\alpha\, a)}\, d\alpha\,. \tag{6}$$

This integral wrt to α can be integrated in the complex plane. But we shall be content to consider the limit case where the width a tends to infinity. Then

$$\Phi_S(\alpha, y) = F_S(\alpha)\, e^{-\alpha\, y}\,, \tag{7}$$

with inverse

$$\phi(x,y) = \frac{2}{\pi}\int_0^\infty \sin(\alpha\, x)\int_0^\infty \sin(\alpha\,\xi)\, f(\xi)\, d\xi\; e^{-\alpha\, y}\, d\alpha\,. \tag{8}$$

Now

$$\begin{aligned}
&\int_0^\infty 2\,\sin(\alpha\, x)\,\sin(\alpha\,\xi)\; e^{-\alpha\, y}\, d\alpha \\
=\;& \int_0^\infty \Big(\cos(\alpha\,(x-\xi)) - \cos(\alpha\,(x+\xi))\Big)\; e^{-\alpha\, y}\, d\alpha \\
=\;& \int_0^\infty \frac{1}{2}\Big(e^{i\,\alpha\,(x-\xi)} + e^{-i\,\alpha\,(x-\xi)} - e^{i\,\alpha\,(x+\xi)} - e^{-i\,\alpha\,(x+\xi)}\Big)\; e^{-\alpha\, y}\, d\alpha \\
=\;& \frac{1}{2}\Big[\frac{e^{-\alpha\, y + i\,\alpha\,(x-\xi)}}{-y + i\,(x-\xi)} + \frac{e^{-\alpha\, y - i\,\alpha\,(x-\xi)}}{-y - i\,(x-\xi)} - \frac{e^{-\alpha\, y + i\,\alpha\,(x+\xi)}}{-y + i\,(x+\xi)} - \frac{e^{-\alpha\, y - i\,\alpha\,(x+\xi)}}{-y - i\,(x+\xi)}\Big]_0^\infty \\
=\;& \frac{y}{y^2 + (x-\xi)^2} - \frac{y}{y^2 + (x+\xi)^2}\,.
\end{aligned} \tag{9}$$

Consequently,

$$\begin{aligned}
\phi(x,y) &= \frac{y}{\pi}\int_0^\infty \Big(\frac{y}{y^2+(x-\xi)^2} - \frac{y}{y^2+(x+\xi)^2}\Big)\, f(\xi)\, d\xi \\
&= \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{sgn}(\xi) f(|\xi|)}{y^2 + (x-\xi)^2}\, d\xi\,.
\end{aligned} \tag{10}$$

The latter writing can be seen as the solution for the half-plane $y \geq 0$ with odd data $f(x)$.

Exercise II.4: **an integral equation and an inverse problem**.

Prove the following formulas, for $x \geq 0$, $b > 0$:

1.
$$\mathcal{F}_C\{e^{-b\,x}\}(\alpha) = \frac{b}{b^2+\alpha^2}\,, \quad \mathcal{F}_S\{e^{-b\,x}\}(\alpha) = \frac{\alpha}{b^2+\alpha^2}\,. \tag{1}$$

2.
$$\int_0^\infty \frac{\cos(v\,x)}{b^2+x^2}\,dv = \frac{\pi}{2\,b}\,e^{-b\,x}\,. \tag{2}$$

3. Solve the integral equation,

$$\int_{-\infty}^\infty \frac{f(u)}{a^2+(x-u)^2}\,du = \frac{1}{b^2+x^2}\,, \quad 0 < a < b\,, \tag{3}$$

for $f(x)$, $x \in]-\infty, \infty[$, and show that the solution reads,

$$f(x) = \frac{1}{\pi}\,\frac{a}{b}\,\frac{b-a}{(b-a)^2+x^2}\,. \tag{4}$$

Proof and solution:

1.
$$\mathcal{F}_C\{e^{-b\,x}\}(\alpha) + i\,\mathcal{F}_S\{e^{-b\,x}\}(\alpha) = \int_0^\infty e^{i\,\alpha\,x-b\,x}\,dx = \frac{-1}{i\,\alpha-b} = \frac{b+i\,\alpha}{b^2+\alpha^2}\,. \tag{5}$$

2. Simply apply the inverse cosine Fourier transform to the first relation,

$$e^{-b\,x} = \frac{2}{\pi}\int_0^\infty \cos(v\,x)\,\frac{b}{b^2+v^2}\,dv\,. \tag{6}$$

3. First,

$$\mathcal{F}\{\frac{1}{b^2+x^2}\}(\alpha) = \int_{-\infty}^\infty \frac{e^{-i\,\alpha\,x}}{b^2+x^2}\,dx = 2\int_0^\infty \frac{\overbrace{\cos(\alpha\,x)}^{\alpha=|\alpha|}}{b^2+x^2}\,dx \overset{\text{eqn (2)}}{=} \frac{\pi}{b}\,e^{-b|\alpha|}\,. \tag{7}$$

Repeated use of this formula yields,

$$\mathcal{F}\{\Big(f(u)*\frac{1}{a^2+u^2}\Big)(x)\}(\alpha) = \begin{cases} \mathcal{F}\{f(x)\}(\alpha)\,\dfrac{\pi}{a}\,e^{-a|\alpha|} & \text{by the convolution theorem} \\ \mathcal{F}\{\displaystyle\int_{-\infty}^\infty \frac{f(u)}{a^2+(x-u)^2}\,du\}(\alpha) & \\ \overset{\text{hyp.}}{=} \mathcal{F}\{\dfrac{1}{b^2+x^2}\}(\alpha) = \dfrac{\pi}{b}\,e^{-b|\alpha|} & \end{cases} \tag{8}$$

Therefore, equating the two lines of the rhs of the latter equation yields,

$$\mathcal{F}\{f(x)\}(\alpha) = \frac{a}{b}\,e^{-(b-a)|\alpha|}\,, \tag{9}$$

whose Fourier inverse is the sought function f,

$$\begin{aligned} f(x) &= \frac{1}{2\,\pi}\int_{-\infty}^{\infty} e^{i\,\alpha\,x}\,\frac{a}{b}\,e^{-(b-a)|\alpha|}\,d\alpha \\ &= \frac{1}{\pi}\,\frac{a}{b}\int_{0}^{\infty} \cos\left(\alpha\,x\right)e^{-(b-a)\alpha}\,d\alpha \\ &= \frac{1}{2\,\pi}\,\frac{a}{b}\int_{0}^{\infty} e^{i\,\alpha\,x-(b-a)\alpha}+e^{-i\,\alpha\,x-(b-a)\alpha}\,d\alpha \\ &= \frac{1}{2\,\pi}\,\frac{a}{b}\Big[\frac{1}{-i\,x+(b-a)}+\frac{1}{i\,x+(b-a)}\Big]. \end{aligned} \tag{10}$$

Exercise II.5: **an elastic half-plane under normal surface load**.

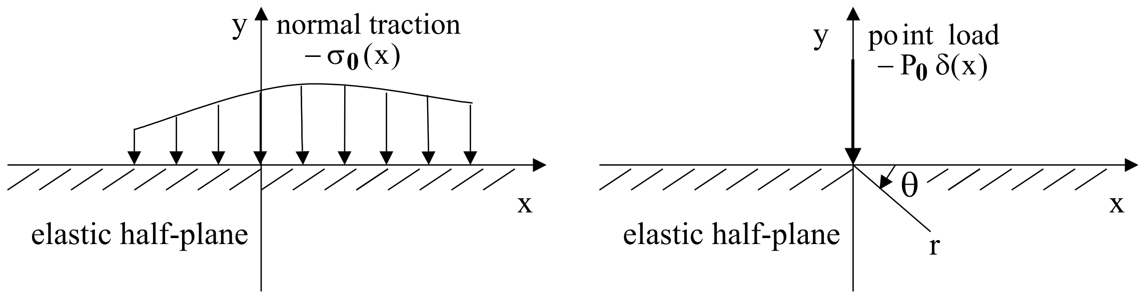

Figure II.7 A half-plane is loaded by a continuous load density, or by a point load, which are normal to the surface.

Some background of plane elasticity has been sketched in Sect. II.6. The whole procedure consists in deriving first the Airy potential $\phi(x, y)$, from which other fields, namely stress and displacement, can be deduced. For the lower half-plane loaded *on* the surface, the Airy potential is governed by the field equation and boundary conditions,

$$\begin{aligned}
&\text{(FE) field equation} && \Delta\Delta\phi = 0, \quad x \in]-\infty, \infty[, \; y < 0; \\
&\text{(BC) boundary conditions} && \sigma_{yy}(x,0) = \frac{\partial^2\phi}{\partial x^2}(x,0) = -\sigma_0(x), \\
& && \sigma_{xy}(x,0) = -\frac{\partial^2\phi}{\partial x \partial y}(x,0) = 0; \\
&\text{(RC) radiation condition} && \phi(x,y) \to 0, \quad \text{as} \quad x^2 + y^2 \to \infty\,.
\end{aligned} \tag{1}$$

1. Derive the potential $\phi(x, y)$,

$$\phi(x,y) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{i\,\alpha\,x + |\alpha|\,y}\,(1 - |\alpha|\,y)\,\frac{\Sigma_0(\alpha)}{\alpha^2}\,d\alpha\,, \tag{2}$$

and deduce the stress components,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \partial^2\phi/\partial y^2 \\ \partial^2\phi/\partial x^2 \\ -\partial^2\phi/\partial x \partial y \end{bmatrix} = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{i\,\alpha\,x + |\alpha|\,y}\,\Sigma_0(\alpha) \begin{bmatrix} -1 - |\alpha|\,y \\ -1 + |\alpha|\,y \\ i\,\alpha\,y \end{bmatrix} d\alpha\,, \tag{3}$$

where

$$\Sigma_0(\alpha) = \int_{-\infty}^{\infty} e^{-i\,\alpha\,x}\,\sigma_0(x)\,dx\,. \tag{4}$$

2. If the load of magnitude P_0 is applied at the point $x = 0$, show that the stresses are

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{2\,P_0}{\pi\,r^4} \begin{bmatrix} x^2\,y \\ y^3 \\ x\,y^2 \end{bmatrix} = \frac{2\,P_0}{\pi\,r} \begin{bmatrix} \cos^2\theta\,\sin\theta \\ \sin^3\theta \\ \cos\theta\,\sin^2\theta \end{bmatrix}. \tag{5}$$

Check that the solution satisfies the boundary conditions.

Solution:

1. Since the space variable x varies between $-\infty$ and $+\infty$, the problem will be solved via the exponential Fourier transform $x \to \alpha$, and we admit that the Fourier transform and partial derivative wrt y operators commute.

The transform of the field equation and boundary conditions,

$$\text{(FE)} \quad \Big(\frac{\partial^2}{\partial y^2} - |\alpha|^2\Big)^2\, \Phi(\alpha, y) = 0, \quad y < 0, \tag{6}$$

solves to

$$\Phi(\alpha, y) = (A(\alpha) + B(\alpha)\, y)\, e^{-|\alpha|\, y} + (C(\alpha) + D(\alpha)\, y)\, e^{|\alpha|\, y} \,. \tag{7}$$

In order to ensure the radiation condition, the Fourier transform itself is required to remain finite at large y, so that $A = B = 0$. The remaining unknowns are defined by the boundary conditions,

$$\begin{aligned} \text{(BC)} \quad -\alpha^2\, \Phi(\alpha, y = 0) &= -\Sigma_0(\alpha) = -\alpha^2\, C(\alpha), \\ -i\,\alpha\, \frac{\partial \Phi}{\partial y}(\alpha, y = 0) &= 0 = -(D + |\alpha|\, C(\alpha))\,, \end{aligned} \tag{8}$$

so that,

$$\Phi(\alpha, y) = e^{|\alpha|\, y}\, (1 - |\alpha|\, y)\, \frac{\Sigma_0(\alpha)}{\alpha^2} \,. \tag{9}$$

Remark

The solution (7) would require some justification. It can be obtained by solving first,

$$\Big(\frac{\partial^2}{\partial y^2} - a^2\Big)\Big(\frac{\partial^2}{\partial y^2} - b^2\Big)\, \Phi(\alpha, y) = 0\,, \tag{10}$$

and next by making $b \to a$.

2. For $\sigma_0(x) = P_0\, \delta(x)$,

$$\Sigma_0(\alpha) = \int_{-\infty}^{\infty} e^{-i\,\alpha\, x}\, \sigma_0(x)\, dx = P_0 \,. \tag{11}$$

Therefore (3) simplifies to

$$\begin{aligned} \sigma_{xx} &= \frac{P_0}{2\,\pi} \int_{-\infty}^{\infty} e^{i\,\alpha\, x + |\alpha|\, y}\, (-1 - |\alpha|\, y)\, d\alpha &&= -\frac{P_0}{\pi} \int_0^{\infty} e^{\alpha\, y}\, \cos(\alpha\, x)\, (1 + \alpha\, y)\, d\alpha \\ \sigma_{yy} &= \frac{P_0}{2\,\pi} \int_{-\infty}^{\infty} e^{i\,\alpha\, x + |\alpha|\, y}\, (-1 + |\alpha|\, y)\, d\alpha &&= -\frac{P_0}{\pi} \int_0^{\infty} e^{\alpha\, y}\, \cos(\alpha\, x)\, (1 - \alpha\, y)\, d\alpha \\ \sigma_{xy} &= \frac{i\, P_0}{2\,\pi} \int_{-\infty}^{\infty} e^{i\,\alpha\, x + |\alpha|\, y}\, \alpha\, y\, d\alpha &&= -\frac{P_0}{\pi} \int_0^{\infty} e^{\alpha\, y}\, \sin(\alpha\, x)\, \alpha\, y\, d\alpha \end{aligned} \tag{12}$$

The integrals can be evaluated by using the first relations of Exercice 4, namely here

$$\int_0^{\infty} e^{\alpha\, y}\, \cos(\alpha\, x)\, d\alpha = \frac{-y}{x^2 + y^2}\,, \tag{13}$$

and therefore

$$\begin{aligned} \int_0^{\infty} e^{\alpha\, y}\, \alpha\, \cos(\alpha\, x)\, d\alpha &= \frac{d}{dy} \int_0^{\infty} e^{\alpha\, y}\, \cos(\alpha\, x)\, d\alpha = \frac{d}{dy}\Big(\frac{-y}{x^2 + y^2}\Big), \\ \int_0^{\infty} e^{\alpha\, y}\, \alpha\, \sin(\alpha\, x)\, d\alpha &= -\frac{d}{dx} \int_0^{\infty} e^{\alpha\, y}\, \cos(\alpha\, x)\, d\alpha = \frac{d}{dx}\Big(\frac{y}{x^2 + y^2}\Big) \,. \end{aligned} \tag{14}$$

The expressions (5) follow for $x = r\cos\theta$, $y = r\sin\theta$.

Moreover, it has been shown in the Chapter devoted to distributions that, in the sense of distributions,

$$\lim_{\epsilon\to 0^+} \frac{2}{\pi}\,\frac{\epsilon^3}{(x^2+\epsilon^2)^2} = \delta(x), \quad \lim_{\epsilon\to 0^+} \frac{2}{\pi}\,\frac{x\,\epsilon^2}{(x^2+\epsilon^2)^2} = 0\,. \tag{15}$$

Therefore, setting here $\epsilon = -y > 0$, the solution (5) is seen to satisfy the boundary conditions $\sigma_{yy}(x,0) = -P_0\,\delta(x)$, and $\sigma_{xy}(x,0) = 0$.

Exercise II.6: **an explicit formula for the Green function**.

Prove the explicit expression of the Green function (II.1.9),

$$u_\delta(x,t) = \frac{1}{\pi}\int_0^\infty \cos(\alpha\, x)\, e^{-\alpha^2 D\, t}\, d\alpha = \frac{1}{2\sqrt{\pi\, D\, t}} \exp\Big(-\frac{x^2}{4\, D\, t}\Big)\,. \tag{1}$$

Proof:

The proof goes as follows. Let us first introduce the change of variable,

$$\alpha \to z = \sqrt{D\, t}\, \alpha\,. \tag{2}$$

Then

$$u_\delta(x,t) = \frac{1}{\pi\sqrt{D\, t}}\, I\Big(\frac{x}{\sqrt{D\, t}}\Big)\,, \tag{3}$$

where

$$I(\mu) \equiv \int_0^\infty e^{-z^2} \cos(\mu\, z)\, dz\,. \tag{4}$$

The intermediate integral I is obtained by forming a differential equation,

$$\begin{aligned} \frac{dI}{d\mu} &= \int_0^\infty (-z\, e^{-z^2}) \sin(\mu\, z)\, dz \\ &= \frac{1}{2}\Big[e^{-z^2} \sin(\mu\, z)\Big]_0^\infty - \frac{\mu}{2}\int_0^\infty e^{-z^2} \cos(\mu\, z)\, dz = -\frac{\mu}{2}\, I(\mu)\,, \end{aligned} \tag{5}$$

that solves to

$$I(\mu) = \frac{\sqrt{\pi}}{2} \exp\Big(-\frac{\mu^2}{4}\Big)\,, \tag{6}$$

admitting $\int_0^\infty e^{-z^2}\, dz = \sqrt{\pi}/2$. □

Exercise II.7: **Inhomogeneous waves over an infinite domain**.

Consider the initial value problem governing the axial displacement $u(x,t)$,

$$\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = h(x,t), \quad t > 0,\ x \in]-\infty, \infty[;\\
&\text{(IC) initial conditions} \quad u(x,0) = f(x); \quad \frac{\partial u}{\partial t}(x,0) = g(x)\,;\\
&\text{(BC) boundary conditions} \quad u(x \to \pm\infty, t) = 0\,,
\end{aligned} \tag{1}$$

in an infinite elastic bar, subject to prescribed initial displacement and velocity fields, $f = f(x)$ and $g = g(x)$ respectively. Here c is speed of elastic waves.

Show that the solution reads,

$$u(x,t) = \frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c\,(t-\tau)}^{x+c\,(t-\tau)} h(y,\tau)\ dy\ d\tau\,. \tag{2}$$

As an alternative to the Fourier transform to be used here, Exercise IV.1 will exploit the method of characteristics.