Chapter I

Solving IBVPs with the Laplace transform

I.1 General perspective

Partial differential equations (PDEs) of mathematical physics [1] are classified in three types, as indicated in the Table below:

Table delineating the three types of PDEs

type	governing equation	unknown u	prototype	appropriate transform
(E) elliptic	$\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$	u(x,y)	potential	Fourier
(P) parabolic	$\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial u}{\partial t}=0$	u(x,t)	heat diffusion	Laplace, Fourier
(H) hyperbolic	$\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial t^2}=0$	u(x,t)	wave propagation	Laplace, Fourier

We will come back later to a more systematic treatment and classification of PDEs.

The one dimensional examples exposed below intend to display some basic features and differences between parabolic and hyperbolic partial differential equations. The constitutive assumptions that lead to the partial differential field equations whose mathematical nature is of prime concern are briefly introduced.

Problems of mathematical physics, continuum mechanics, fluid mechanics, strength of materials, hydrology, thermal diffusion $\cdots$, can be cast as IBVPs:

IBVPs: **Initial and Boundary Value Problems**

We will take care to systematically define problems in that broad framework. To introduce the ideas, we delineate three types of relations, as follows.

I.1.1 (FE) Field Equations

The field equations that govern the problems are partial differential equations in space and time. As indicated just above, they can be classified as elliptic, parabolic and hyperbolic. This

[1] Posted November 22, 2008; Updated, March 26, 2009

chapter will consider prototypes of the two later types, and emphasize their physical meaning, interpretation and fundamental differences.

We can not stress enough that

(P) for a parabolic equation, the information diffuses at infinite speed, and progressively,

while

(H) for a hyperbolic equation, the information propagates at finite speed and discontinuously.

These field equations should be satisfied *within the body*, say Ω. They are not required to hold on the boundary $\partial\Omega$. Ω is viewed as an *open* set of points in space, in the topological sense.

In this chapter, the solutions $u(x,t)$ are obtained through the Laplace transform in time,

$$u(x,t) \to U(x,p) = \mathcal{L}\{u(x,t)\}(p), \tag{I.1.1}$$

and we shamelessly admit that the Laplace transform and partial derivative in space operators commute,

$$\frac{\partial}{\partial x}\mathcal{L}\{u(x,t)\}(p) = \mathcal{L}\{\frac{\partial}{\partial x}u(x,t)\}(p)\,. \tag{I.1.2}$$

In contrast, we will see that the use of the Fourier transform adopts the dual rule, in as far as the Fourier variable is the space variable x,

$$u(x,t) \to U(\alpha,t) = \mathcal{F}\{u(x,t)\}(\alpha)\,. \tag{I.1.3}$$

A qualitative motivation consists in establishing a correspondence between the time variable, a positive quantity, and the definition of the (one-sided) Laplace transform which involves an integration over a positive variable.

In contrast, the Fourier transform involves the integration over the whole real set, which is interpreted as a spatial axis.

Attention should be paid to the interpretation of the physical phenomena considered. For example, hyperbolic problems in continuum mechanics involves the speed of propagation of elastic waves. This wave speed should not be confused with the velocity of particles. To understand the difference, consider a wave moving over a fluid surface. When we follow the phenomenon, we attach our eyes to the top of the wave. At two subsequent times, the particles at the top of the waves are not the same, and therefore the wave speed and particle velocity are two distinct functions of space and time. In fact, in the examples to be considered, the wave speeds are constant in space and time.

I.1.2 (IC) Initial Conditions

Initial Conditions come into picture when the physical time is involved, namely in (P) and (H) equations.

Roughly, the number of initial conditions depends on the order of the partial derivative(s) wrt time in the field equation (FE). For example, one initial condition is required for the heat diffusion problem, and two for the wave propagation problem.

I.1.3 (BC) Boundary Conditions

Similarly, the number of boundary conditions depends on the order of the partial derivative(s) wrt space in the field equation. For example, for the bending of a beam, four conditions are required since the field equation involves a fourth order derivative in space.

This rule applies when the body is finite. The treatment of semi-infinite or infinite bodies is both easier and more delicate, and often requires some knowledge-based decisions, which still are quite easy to enter. Typically, since the boundary is rejected to infinity, the boundary condition is replaced by either an asymptotic requirement, e.g. the solution should remain finite at infinity $\cdots$, or by a *radiation condition*. For the standard problems considered here, the source of disturbance is located at finite distance, and the radiation condition ensures that the information diffuses/propagates toward infinity. In contrast, for a finite body, waves would be reflected by the boundary, at least partially, back to the body. Alternatively, one could also consider diffraction problems, where the source of the disturbance emanates from infinity.

Boundary conditions are phrased in terms of the main unknown, or its time or space derivative(s).

For example, for an elastic problem, the displacement may be prescribed at a boundary point: its gradient (strain) should be seen as participating to the response of the structure. The velocity may be substituted for the displacement. Conversely, instead of prescribing the displacement, one may prescribe its gradient: then the displacement participates to the response of the structure.

Attention should be paid not to prescribe incompatible boundary conditions: e.g. for a thermal diffusion problem, either the temperature or the heat flux (temperature gradient) may be prescribed, but *not both* simultaneously.

The situation is more complex for higher order problems, e.g. for the bending of beams. Of course, any boundary condition is definitely motivated by the underlying physics.

It is time now to turn to examples.

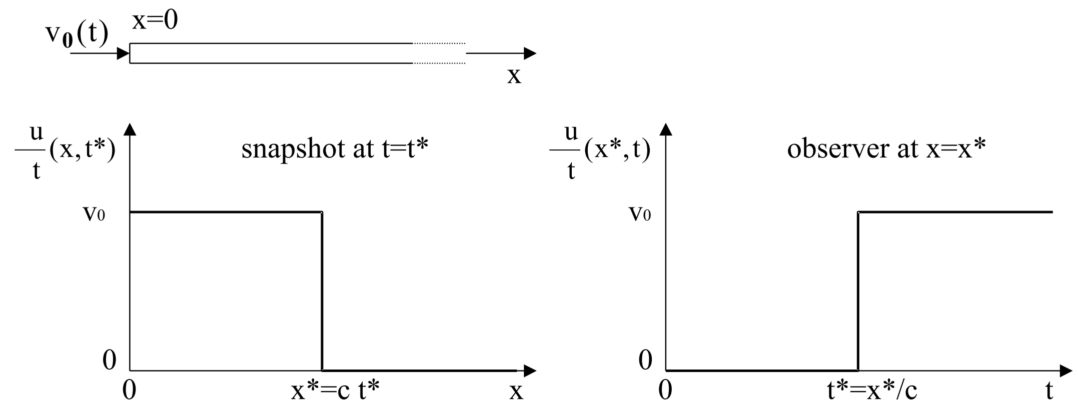

Figure I.1 A semi-infinite elastic bar is subject to a velocity discontinuity. Spatial and time profiles of the particle velocity. The discontinuity propagates along the bar at the wave speed $c = \sqrt{E/\rho}$ where $E > 0$ is Young's modulus and $\rho > 0$ the mass density.

I.2 An example of hyperbolic PDE: propagation of a shock wave

A semi-infinite elastic plane, or half-space, is subject to a load normal to its boundary. The motion is one dimensional and could also be viewed as due to an axial load on the end of an elastic bar with vanishing Poisson's ratio. The bar is initially at rest, and the loading takes the form of an arbitrary velocity discontinuity $v = v_0(t)$.

The issue is to derive the axial displacement $u(x,t)$ and axial velocity $v(x,t)$ of the points of the bar while the mechanical information propagates along the bar with a wave speed,

$$c = \sqrt{\frac{E}{\rho}}, \tag{I.2.1}$$

where $E > 0$ and $\rho > 0$ are respectively the Young's modulus and mass density of the material.

The governing equations of dynamic linear elasticity are,

$$\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad t > 0,\ x > 0;\\
&\text{(IC) initial conditions} \quad u(x,0) = 0; \quad \frac{\partial u}{\partial t}(x,0) = 0\,;\\
&\text{(BC) boundary condition} \quad \frac{\partial u}{\partial t}(0,t) = v_0(t); \quad \text{radiation condition (RC)}\,.
\end{aligned} \tag{I.2.2}$$

The field equation is obtained by combining

$$\begin{aligned}
&\text{momentum balance} \quad \frac{\partial \sigma}{\partial x} - \rho\,\frac{\partial^2 u}{\partial x^2} = 0;\\
&\text{elasticity} \quad \sigma = E\,\frac{\partial u}{\partial x}\,,
\end{aligned} \tag{I.2.3}$$

where σ is the axial stress.

The first initial condition (IC) means that the displacement is measured from time $t = 0$, or said otherwise, that the configuration (geometry) at time $t = 0$ is used as a reference. The second (IC) simply means that the bar is initially at rest.

The radiation condition (RC) is intended to imply that the mechanical information propagates in a single direction, and that the bar is either of infinite length, or, at least, that the signal has not the time to reach the right boundary of the bar in the time window of interest. Indeed, any function of the form $f_1(x - c\,t) + f_2(x + c\,t)$ satisfies the field equation. A function of $x - c\,t$ represents a signal that propagates toward increasing x. To see this, let us keep the eyes on some given value of f_1, corresponding to $x - c\,t$ equal to some constant. Then clearly, sine the wave speed c is a positive quantity, the point we follow moves in time toward increasing x.

The solution is obtained through the Laplace transform in time,

$$u(x,t) \to U(x,p) = \mathcal{L}\{u(x,t)\}(p), \tag{I.2.4}$$

and we admit that the Laplace transform and partial derivative in space operators commute,

$$\frac{\partial}{\partial x}\mathcal{L}\{u(x,t)\}(p) = \mathcal{L}\{\frac{\partial}{\partial x}u(x,t)\}(p)\,. \tag{I.2.5}$$

Therefore,

$$\begin{aligned}
&\text{(FE)} \quad \frac{\partial^2 U(x,p)}{\partial x^2} - \frac{1}{c^2}\Big(p^2\,U(x,p) - p\,\underbrace{u(x,0)}_{=0,\ \text{(IC)}} - \underbrace{\frac{\partial u}{\partial t}(x,0)}_{=0,\ \text{(IC)}}\Big) = 0\\
&\Rightarrow \quad U(x,p) = a(p)\,\exp(-\frac{p}{c}x) + \underbrace{b(p)\,\exp(+\frac{p}{c}x)}_{b(p)=0,\ \text{(RC)}}\,.
\end{aligned} \tag{I.2.6}$$

The term $\exp(-p\,x/c)$ gives rise to a wave which propagates toward increasing value of x, and the term $\exp(p\,x/c)$ gives rise to a wave which propagates toward decreasing value of x. Where do these assertions come from? There is no direct answer, simply they can be checked on the result to be obtained. Thus the radiation condition implies to set $b(p)$ equal to 0.

In turn, taking the Laplace transform of the (BC),

$$\begin{aligned}
&\text{(BC)}\; p\,U(0,p) - \underbrace{u(0,0)}_{=0,\ \text{(IC)}} = \mathcal{L}\{v_0(t)\}(p) \\
&\Rightarrow \quad U(x,p) = \frac{1}{p}\exp(-\frac{p}{c}x)\,\mathcal{L}\{v_0(t\}(p)\,.
\end{aligned} \tag{I.2.7}$$

The inverse Laplace transform is a convolution integral,

$$u(x,t) = \int_0^t \mathcal{H}(t-\frac{x}{c}-\tau)\,v_0(\tau)\,d\tau\,, \tag{I.2.8}$$

which, for a shock $v_0(t) = V_0\,\mathcal{H}(t)$, simplifies to

$$u(x,t) = V_0\,(t-\frac{x}{c})\,\mathcal{H}(t-\frac{x}{c}), \quad \frac{\partial u}{\partial t}(x,t) = V_0\,\mathcal{H}(t-\frac{x}{c})\,. \tag{I.2.9}$$

The analysis of the velocity of the particles reveals two main characteristics of a partial differential equation of the hyperbolic type in non dissipative materials, Fig. I.1:

(H1). **the mechanical information propagates at finite speed**, namely c, which is therefore termed elastic wave speed;

(H2). **the wave front carrying the mechanical information propagates** undistorted and **with a constant amplitude**.

Besides, this example highlights the respective meanings of elastic wave speed c and velocity of the particles $\partial u/\partial t(x,t)$.

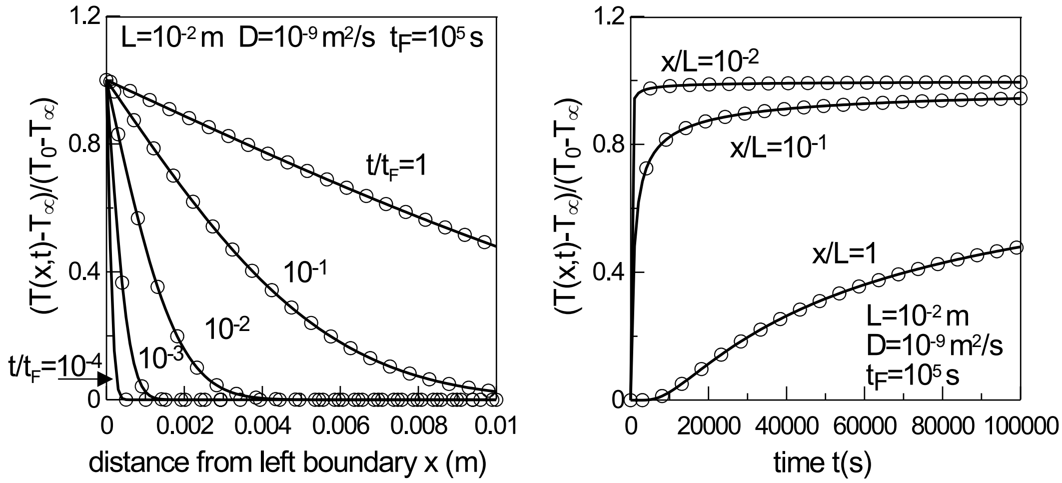

Figure I.2 Heat diffusion on a semi-infinite bar subject to a heat shock at its left boundary $x = 0$. The characteristic time t_F that can be used to describe the information that reaches the position $x = L$ is equal to L^2/D.

I.3 A parabolic PDE: diffusion of a heat shock

As a second example of PDE, let us consider the diffusion of a heat shock in a semi-infinite body.

A semi-infinite plane, or half-space, or bar, is subject to a given temperature $T = T_0(t)$ at its left boundary $x = 0$. The thermal diffusion is one dimensional. The initial temperature along the bar is uniform, $T(x,t) = T_\infty$. It is more convenient to work with the field $\theta(x,t) = T(x,t) - T_\infty$ than with the temperature itself.

For a rigid material, in absence of heat source, the energy equation links the divergence of the heat flux $\mathbf{Q}$ [unit: kg/s^3], and the time rate of the temperature field $\theta(x,t)$, namely

$$\text{energy equation} \quad \operatorname{div}\mathbf{Q} + C\,\frac{\partial \theta}{\partial t} = 0\,, \tag{I.3.1}$$

or in cartesian axes and using the convention of summation over repeated indices (the index i varies from 1 to n in a space of dimension n),

$$\frac{\partial Q_i}{\partial x_i} + C\,\frac{\partial \theta}{\partial t} = 0\,. \tag{I.3.2}$$

Here $C >$ [unit: kg/m/s^2/°K] is the heat capacity per unit volume. The heat flux is related to the temperature gradient by Fourier law

$$\text{Fourier law} \quad \mathbf{Q} = -k_T\,\nabla\theta\,, \tag{I.3.3}$$

or componentwise,

$$Q_i = -k_T\,\frac{\partial \theta}{\partial x_i}\,, \tag{I.3.4}$$

where $k_T > 0$ [unit: kg×m/s^3/°K] is the thermal conductivity.

Inserting Fourier's law in the energy equation shows that a single material coefficient D [unit: m^2/s],

$$D = \frac{k_T}{C} > 0, \tag{I.3.5}$$

that we shall term diffusivity, appears in the field equation.

The initial and boundary value problem (IBVP) is thus governed by the following set of equations:

$$\begin{aligned}
&\text{field equation (FE)} \quad D\,\frac{\partial^2\theta}{\partial x^2} - \frac{\partial \theta}{\partial t} = 0, \quad t>0,\ x>0;\\
&\text{initial condition (IC)} \quad \theta(x,0) = 0\,;\\
&\text{boundary condition (BC)} \quad \theta(0,t) = \theta_0(t); \quad \text{radiation condition (RC)}\,.
\end{aligned} \tag{I.3.6}$$

The radiation condition is intended to imply that the thermal information diffuses in a single direction, namely toward increasing x, and that the bar is either of infinite length.

The solution is obtained through the Laplace transform

$$\theta(x,t) \to \Theta(x,p) = \mathcal{L}\{\theta(x,t)\}(p)\,. \tag{I.3.7}$$

Therefore,

$$
\begin{aligned}
&\text{(FE)}\ D\,\frac{\partial^2\Theta(x,p)}{\partial x^2} - \Big(p\,\Theta(x,p) - \underbrace{\theta(x,0)}_{=0,\ \text{(IC)}}\Big) = 0,\\
&\Rightarrow\quad \Theta(x,p) = a(p)\,\exp(-\sqrt{\frac{p}{D}}\,x) + \underbrace{b(p)\,\exp(\sqrt{\frac{p}{D}}\,x)}_{b(p)=0,\ \text{(RC)}}\\
&\text{(BC)}\ \Theta(0,p) = \mathcal{L}\{\theta_0(t\}(p)\\
&\Rightarrow\quad \Theta(x,p) = \mathcal{L}\{\theta_0(t\}(p)\,\exp(-\sqrt{\frac{p}{D}}\,x)\,.
\end{aligned}
\tag{I.3.8}
$$

The multiform complex function $\sqrt{p}$ has been made uniform by introducing a branch cut along the negative axis $\Re p \le 0$, and by defining $p = |p|\,\exp(i\,\theta)$ with $\theta \in]-\pi,\pi]$, and $\sqrt{p} = \sqrt{|p|}\,\exp(i\,\theta/2)$ so that $\Re\sqrt{p} \ge 0$. Then the term $\exp(\sqrt{p/D}\,x)$ gives rise to heat propagation toward decreasing x, in contradiction with the radiation condition, and this justifies why $b(p)$ has been forced to vanish.

The inverse transform is a convolution product,

$$
\theta(x,t) = \int_0^t \theta_0(t-u)\ \mathcal{L}^{-1}\{\exp\Big(-x\sqrt{\frac{p}{D}}\Big)\}(u)\,du\,. \tag{I.3.9}
$$

Using the relation, established later in Exercise I.6,

$$
\mathcal{L}^{-1}\{\exp\Big(-x\sqrt{\frac{p}{D}}\Big)\}(u) = \frac{x}{2\sqrt{\pi\,D\,u^3}}\ \exp\Big(-\frac{x^2}{4\,Du}\Big), \tag{I.3.10}
$$

then

$$
\theta(x,t) = \frac{x}{2\sqrt{D\pi}} \int_0^t \theta_0(t-u)\ \frac{1}{u^{3/2}}\ \exp\Big(-\frac{x^2}{4\,Du}\Big)\ du\,. \tag{I.3.11}
$$

With the further change of variable

$$
u \to v \quad \text{with} \quad v^2 = \frac{x^2}{4D\,u}, \tag{I.3.12}
$$

the inverse Laplace transform takes the form,

$$
\theta(x,t) = \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{D\,t}}^{\infty} \theta_0(t - \frac{x^2}{4\,D\,v^2})\ \exp(-v^2)\,dv\,, \tag{I.3.13}
$$

which, for a shock $\theta_0(t) = (T_0 - T_\infty)\,\mathcal{H}(t)$, simplifies to

$$
T(x,t) - T_\infty = (T_0 - T_\infty)\ \text{erfc}\Big(\frac{x}{2\sqrt{D\,t}}\Big)\,. \tag{I.3.14}
$$

The plot of the evolution of the spatial profiles of the temperature reveals two main characteristics of a partial differential equation of the parabolic type, Fig. I.2:

(P1). **the thermal information diffuses at infinite speed**: indeed, the fact that the boundary $x = 0$ has been heated is known instantaneously at any point of the bar;

(P2). however, **the amplitude of the thermal shock applied at the boundary requires time to fully develop** in the bar. In fact, the temperature needs an infinite time to equilibrate along the bar.

Note however, that a modification of the Fourier's law, or of the energy equation, which goes by the name of Cataneo, provides finite propagation speeds. The point is not considered further here.

In contrast to wave propagation, where time and space are involved in linear expressions $x \pm c\,t$, here the space variable is associated with the square root of the time variable. Therefore diffusion over a distance $2\,L$ requires a time interval four times larger than the time interval of diffusion over a length L.

The error and complementary error functions

Use has been made of the error function

$$\mathrm{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-v^2}\, dv\,, \tag{I.3.15}$$

and of the complementary error function

$$\mathrm{erfc}(y) = \frac{2}{\sqrt{\pi}} \int_y^\infty e^{-v^2}\, dv\,. \tag{I.3.16}$$

Note the relations

$$\mathrm{erf}(y) + \mathrm{erfc}(y) = 1, \quad \mathrm{erf}(-y) = -\mathrm{erf}(y), \quad \mathrm{erfc}(-y) = 2 - \mathrm{erfc}(y)\,, \tag{I.3.17}$$

and the particular values $\mathrm{erfc}(0) = 1$, $\mathrm{erfc}(\infty) = 0$.

I.4 An equation displaying advection-diffusion

The diffusion of a species dissolved in a fluid at rest obeys the same field equation as thermal diffusion. Diffusion takes place so as to homogenize the concentration $c = c(x,t)$ of a solute in space. Usually however, the fluid itself moves due to different physical phenomena: for example, seepage of the fluid through a porous medium is triggered by a gradient of fluid pressure and governed by Darcy law. Let us assume the velocity v of the fluid, referred to as *advective velocity*, to be a given constant.

Thus, we assume the fluid to move with velocity v and the solute with velocity v_s. In order to highlight that diffusion is relative to the fluid, two fluxes are introduced, a diffusive flux $\mathbf{J}^d$ and an absolute flux $\mathbf{J}$,

$$\underbrace{\mathbf{J} = c\,v_s}_{\text{absolute flux}}\ , \quad \underbrace{\mathbf{J}^d = c\,(v_s - v)}_{\text{diffusive flux}}\,, \tag{I.4.1}$$

whence,

$$\mathbf{J} = \mathbf{J}^d + c\,v\,. \tag{I.4.2}$$

The diffusion phenomenon is governed by Fick's law that relates the *diffusive* flux $\mathbf{J}^d = c\,(v_s - v)$ to the gradient of concentration via a coefficient of molecular diffusion D [unit: m^2/s],

$$\text{Fick's law} \quad \mathbf{J}^d = -D\,\nabla c\,, \tag{I.4.3}$$

and by the mass balance, which, in terms of concentration and *absolute* flux $\mathbf{J} = c\,v_s$, writes,

$$\text{balance of mass} \quad \frac{\partial c}{\partial t} + \mathrm{div}\,\mathbf{J} = 0\,. \tag{I.4.4}$$

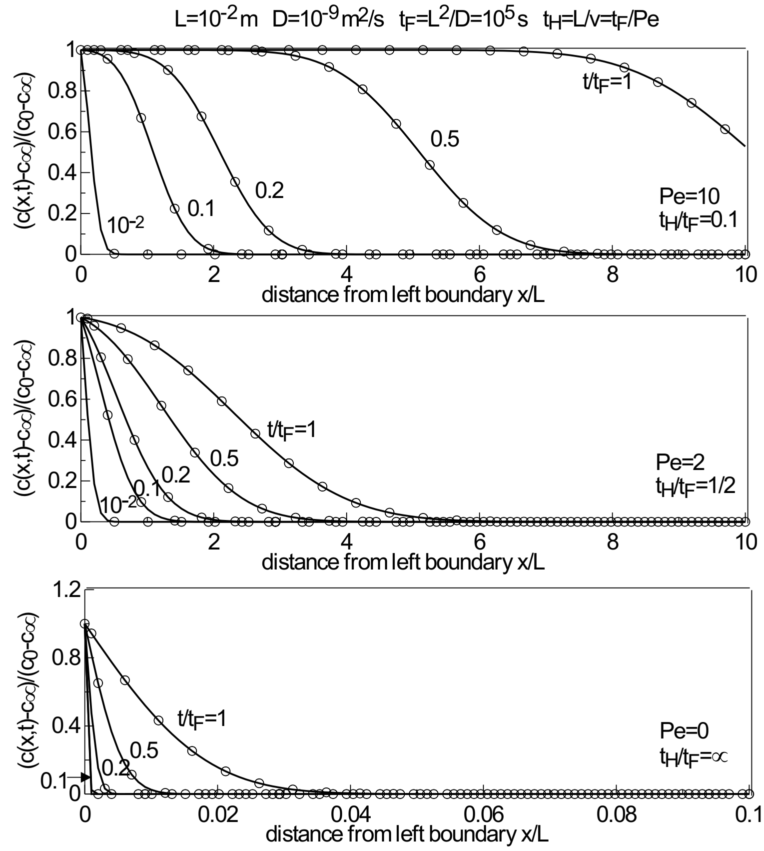

Figure I.3 Advection-diffusion along a semi-infinite bar of a species whose concentration is subject at time $t = 0$ to a sudden increase at the left boundary $x = 0$. The fluid is animated with a velocity v such that the Péclet number Pe is equal to 10, 2 and 0 (pure diffusion) respectively. Focus is on the events that take place at point $x = L$. The time which characterizes the propagation phenomenon is $t_H = L/v$ while the characteristic time of diffusion is $t_F = L^2/D$, with D the diffusion coefficient. Therefore $\mathrm{Pe} = L\,v/D = t_H/t_F$.

The equations of diffusion analyzed in Sect. I.3 modify to

$$
\begin{aligned}
&\text{(FE) field equation} \quad \overbrace{D\,\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial t}}^{\text{diffusive terms}} = \overbrace{v\,\frac{\partial c}{\partial x}}^{\text{advective term}}, \quad t > 0,\ x > 0; \\
&\text{(IC) initial condition} \quad c(x,0) = c_i\,; \\
&\text{(BC) boundary condition} \quad c(0,t) = c_0(t); \quad \text{and (RC) a radiation condition}\,.
\end{aligned}
\tag{I.4.5}
$$

A concentration discontinuity is imposed at $x = 0$, namely $c(0,t) = c_0\,\mathcal{H}(t)$.

The solution $c(x,t)$ is obtained through the Laplace transform $c(x,t) \to C(x,p)$. First, the field equation becomes an ordinary differential equation wrt space where the Laplace variable

p is viewed as a parameter,

$$\text{(FE)}\quad D\,\frac{d^2\tilde{C}}{dx^2} - v\,\frac{d\tilde{C}}{dx} - p\,\tilde{C} = 0, \quad \tilde{C}(x,p) \equiv C(x,p) - \frac{c_i}{p}\,. \tag{I.4.6}$$

The solution,

$$\tilde{C}(x,p) = a(p)\,\exp\Big(\frac{v\,x}{2D} - \sqrt{(\frac{v}{2D})^2 + \frac{p}{D}}\,x\Big) + b(p)\,\exp\Big(\frac{v\,x}{2D} + \sqrt{(\frac{v}{2D})^2 + \frac{p}{D}}\,x\Big)\,, \tag{I.4.7}$$

involves two unknowns $a(p)$ and $b(p)$. The unknown $b(p)$ is set to 0, because it multiplies a function that would give rise to diffusion from right to left: the radiation condition (RC) intends to prevent this phenomenon. The second unknown $a(p)$ results from the (BC):

$$\text{(RC)}\quad b(p) = 0, \quad \text{(BC)}\quad a(p) = \frac{c_0 - c_i}{p}\,. \tag{I.4.8}$$

The resulting complete solution in the Laplace domain

$$C(x,p) = \frac{c_i}{p} + \frac{c_0 - c_i}{p}\,\exp\Big(-x\,\sqrt{(\frac{v}{2D})^2 + \frac{p}{D}}\Big)\,\exp\Big(\frac{v\,x}{2D}\Big)\,, \tag{I.4.9}$$

may be slightly transformed to

$$C(x,P) = \frac{c_i}{P-\alpha} + \frac{c_0 - c_i}{P-\alpha}\,\exp\Big(-x\,\sqrt{\frac{P}{D}}\Big)\,\exp\Big(\frac{v\,x}{2D}\Big), \quad P \equiv p + \alpha, \quad \alpha \equiv \frac{v^2}{4\,D}\,. \tag{I.4.10}$$

The inverse transform of the second term is established in Exercise I.7,

$$\begin{aligned}&\mathcal{L}^{-1}\{\frac{\exp(-x\,\sqrt{P/D})}{P-\alpha}\}(t)\\ &= \frac{\exp(\alpha\,t)}{2}\Big(\exp\,(x\,\sqrt{\frac{\alpha}{D}})\,\text{erfc}(\frac{x}{2\sqrt{Dt}} + \sqrt{\alpha\,t}) + \exp\,(-x\,\sqrt{\frac{\alpha}{D}})\,\text{erfc}(\frac{x}{2\sqrt{Dt}} - \sqrt{\alpha\,t})\Big)\,.\end{aligned} \tag{I.4.11}$$

Whence, in view of the rule,

$$\mathcal{L}\{c(x,t)\}(P) = \mathcal{L}\{e^{-\alpha\,t}\,c(x,t)\}(p = P - \alpha)\,, \tag{I.4.12}$$

the solution can finally be cast in the format, which holds for positive or negative velocity v,

$$c(x,t) = c_i + \frac{1}{2}\,(c_0 - c_i)\,\Big(\,\text{erfc}\Big(\frac{x - v\,t}{2\sqrt{D\,t}}\Big) + \exp(\frac{v\,x}{D})\,\text{erfc}\Big(\frac{x + v\,t}{2\sqrt{D\,t}}\Big)\,\Big)\,. \tag{I.4.13}$$

Note that any arbitrary function $f = f(x - v\,t)$ leaves unchanged the first order part of the field equation (I.4.5). The solution can thus be seen as displaying a front propagating toward $x = \infty$, which is smoothed out by the diffusion phenomenon. The dimensionless Péclet number Pe quantifies the relative weight of advection and diffusion,

$$\text{Pe} = \frac{L\,v}{D}\begin{cases} \ll 1 & \text{diffusion dominated flow}\\ \gg 1 & \text{advection dominated flow}\end{cases} \tag{I.4.14}$$

The length L is a characteristic length of the problem, e.g. mean grain size in granular media, or length of the column for breakthrough tests in a column of finite length.

For transport of species, the Péclet number represents the ratio of the number, or mass, of particles transported by advection and diffusion. For heat transport, it gives an indication of the ratio of the heat transported by advection and by conduction.

Exercise I.1: **Playing with long darts**.

A dart, moving at uniform horizontal velocity $-v_0$, is headed toward a vertical wall, located at the position $x = 0$. Its head hits the wall at time $t = 0$, and thereafter remains glued to the wall. Three snapshots have been taken, at different times, and displayed on Fig. I.5.

The dart is long enough, so that during the time interval of interest, no wave reaches its right end. In other words, for the present purpose, the dart can be considered as semi-infinite.

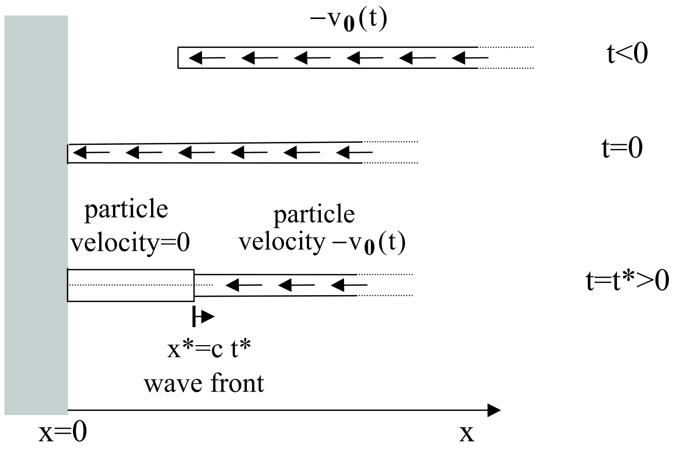

Figure I.4 An elastic dart moving at speed $-v_0$ hits a rigid target at time $t = 0$. The shock then propagates along the dart at the speed of elastic longitudinal waves. The part of the dart behind the wave front is set to rest and it undergoes axial compression: although the analysis here is one-dimensional, we have visualized this aspect by a (mechanically realistic) lateral expansion.

The dart is assumed to be made in a linear elastic material, in which the longitudinal waves propagate at speed c. The equations of dynamic linear elasticity governing the axial displacement $u(x, t)$ of points of the dart are,

$$
\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad t > 0,\ x > 0;\\
&\text{(IC) initial conditions} \quad u(x, 0) = 0; \quad \frac{\partial u}{\partial t}(x, 0) = -v_0\,;\\
&\text{(BC) boundary condition} \quad u(0, t) = 0; \quad \text{radiation condition (RC)} \ \frac{\partial u}{\partial x}(\infty, t) = 0\,.
\end{aligned}
\tag{1}
$$

Find the displacement $u(x, t)$ and give a vivid interpretation of the event.

Solution:

The solution is obtained through the Laplace transform in time,

$$
u(x, t) \to U(x, p) = \mathcal{L}\{u(x, t)\}(p), \tag{2}
$$

and we admit that the Laplace transform and partial derivative in space operators commute,

$$
\frac{\partial}{\partial x}\mathcal{L}\{u(x, t)\}(p) = \mathcal{L}\{\frac{\partial}{\partial x}u(x, t)\}(p)\,. \tag{3}
$$

Therefore,

$$\begin{aligned}
&\text{(FE)} \quad \frac{\partial^2 U(x,p)}{\partial x^2} - \frac{1}{c^2}\Big(p^2\, U(x,p) - p\, \underbrace{u(x,0)}_{=0,\ \text{(IC)}} - \underbrace{\frac{\partial u}{\partial t}(x,0)}_{=-v_0,\ \text{(IC)}}\Big) = 0 \\
&\frac{d^2}{dx^2}\Big(U(x,p) + \frac{v_0}{p^2}\Big) = \frac{p^2}{c^2}\Big(U(x,p) + \frac{v_0}{p^2}\Big)\,.
\end{aligned} \tag{4}$$

The change of notation, from partial to total derivative wrt space, is intended to convey the idea that the second relation is seen as an ordinary differential equation in space, where the Laplace variable plays the role of a parameter. Then

$$U(x,p) + \frac{v_0}{p^2} = a(p)\, \exp(\frac{p}{c}x) + b(p)\, \exp(-\frac{p}{c}x)\,. \tag{5}$$

The first term in the solution above would give rise to a wave moving to left, which is prevented by the radiation condition:

$$\begin{aligned}
&\text{(RC)} \quad a(p) = 0 \\
&\text{(BC)} \quad 0 + \frac{v_0}{p^2} = b(p)\, \exp(0) \\
&U(x,p) = \frac{v_0}{p^2}\Big(\exp(-\frac{p}{c}x) - 1\Big)\,.
\end{aligned} \tag{6}$$

Inverse Laplace transform yields in turn the displacement $u(x,t)$,

$$u(x,t) = v_0\Big((t - \frac{x}{c})\,\mathcal{H}(t - \frac{x}{c}) - t\,\mathcal{H}(t)\Big) = \begin{cases} -v_0\,\dfrac{x}{c}, & x < c\,t; \\ -v_0\,t, & x \geq c\,t; \end{cases} \tag{7}$$

the particle velocity,

$$\frac{\partial u(x,t)}{\partial t} = \begin{cases} 0, & x < c\,t; \\ -v_0, & x > c\,t; \end{cases} \tag{8}$$

and the strain,

$$\frac{\partial u(x,t)}{\partial x} = \begin{cases} -\dfrac{v_0}{c}, & x < c\,t; \\ 0, & x > c\,t\,. \end{cases} \tag{9}$$

These relations clearly indicate that the mechanical information “ the dart head is glued to the wall” propagates to the right along the dart at speed c. At a given time t^*, only points sufficiently close to the wall have received the information, while further points still move with the initial speed $-v_0$. Points behind the wave front undergo compressive straining, while the part of the dart to the right of the wave front is undeformed yet. An observer, located at x^* needs to wait a time $t = x^*/c$ to receive the information: the velocity of this point then stops immediately, and completely.

Exercise I.2: **Laplace transforms of periodic functions**.

Consider the three periodic functions sketched on Fig. I.5.

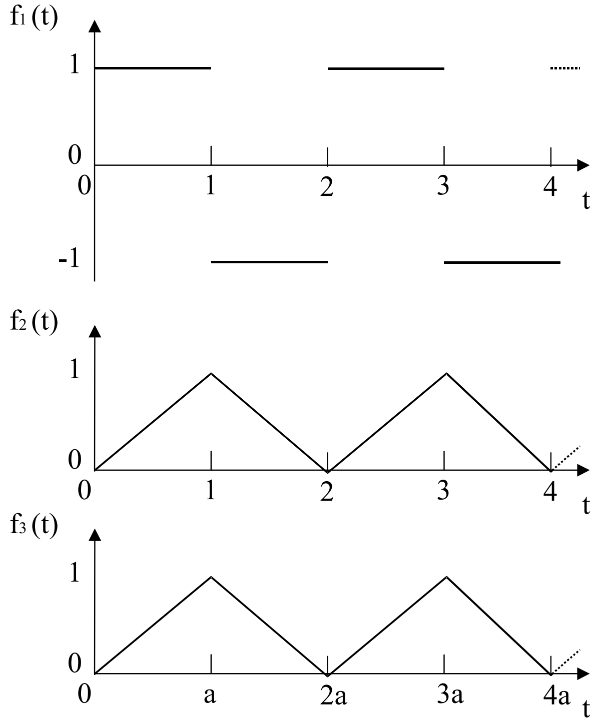

Figure I.5 Periodic functions; for $f_3(t)$, the real $a > 0$ is strictly positive.

Calculate their Laplace transforms:

$$\mathcal{L}\{f_1(t)\}(p) = \frac{1}{p}\,\tanh\Big(\frac{p}{2}\Big);\quad \mathcal{L}\{f_2(t)\}(p) = \frac{1}{p^2}\,\tanh\Big(\frac{p}{2}\Big);\quad \mathcal{L}\{f_3(t)\}(p) = \frac{1}{a\,p^2}\,\tanh\Big(\frac{a\,p}{2}\Big)\,. \tag{1}$$

Proof:

1. These functions are periodic, for $t > 0$, with period $T = 2$:

$$\mathcal{L}\{f_1(t)\}(p) = \frac{1}{1-e^{-2p}}\int_0^2 e^{-t\,p}\,f_1(t)\,dt = \frac{1}{1-e^{-2p}}\,\Big(\int_0^1 e^{-t\,p}\,dt + \int_1^2 e^{-t\,p}\,(-1)\,dt\Big)\,. \tag{2}$$

2. The function $f_2(t)$ is the integral of $f_1(t)$:

$$\mathcal{L}\{f_2(t)\}(p) = \mathcal{L}\{\int_0^t f_1(t)\}(p) = \frac{1}{p}\,\mathcal{L}(f_1(t))(p)\,. \tag{3}$$

3. Since $f_3(t) = f_2(b\,t)$, with $b = 1/a$,

$$\mathcal{L}\{f_3(t)\}(p) = \frac{1}{b}\,\mathcal{L}\{f_2(t))(\frac{p}{b})\,. \tag{4}$$

Exercise I.3: **Longitudinal vibrations of a finite bar**.

A bar, of finite length L, is fixed at its left boundary $x = 0$. It is at rest for times $t < 0$. At $t = 0$, it is submitted to a force F_0 at its right boundary $x = L$.

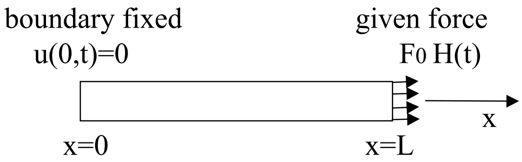

Figure I.6 An elastic bar, fixed at its left boundary, is hit by a sudden load at its right boundary at time t=0.

The bar is made of a linear elastic material, with a Young's modulus E and a section S, and the longitudinal waves propagate at speed c.

The equations of dynamic linear elasticity governing the axial displacement $u(x,t)$ of the points inside the bar are,

$$\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad t > 0,\ x \in]0, L[;\\
&\text{(IC) initial conditions} \quad u(x,0) = 0; \quad \frac{\partial u}{\partial t}(x,0) = 0\,;\\
&\text{(BC) boundary conditions} \quad u(0,t) = 0; \quad \frac{\partial u}{\partial x}(L,t) = \frac{F_0}{ES}\,\mathcal{H}(t)\,.
\end{aligned} \tag{1}$$

Find the displacement $u(L,t)$ of the right boundary and give a vivid interpretation of the phenomenon.

Solution:

The solution is obtained through the Laplace transform in time,

$$u(x,t) \to U(x,p) = \mathcal{L}\{u(x,t)\}(p), \tag{2}$$

and we admit that the Laplace transform and partial derivative in space operators commute,

$$\frac{\partial}{\partial x}\mathcal{L}\{u(x,t)\}(p) = \mathcal{L}\{\frac{\partial}{\partial x}u(x,t)\}(p)\,. \tag{3}$$

Therefore,

$$\begin{aligned}
&\text{(FE)} \quad \frac{\partial^2 U(x,p)}{\partial x^2} - \frac{1}{c^2}\Big(p^2\,U(x,p) - p\,\underbrace{u(x,0)}_{=0,\ \text{(IC)}} - \underbrace{\frac{\partial u}{\partial t}(x,0)}_{=0,\ \text{(IC)}}\Big) = 0\\
&\qquad \frac{d^2}{dx^2}U(x,p) = \frac{p^2}{c^2}\,U(x,p)\,.
\end{aligned} \tag{4}$$

The change of notation, from partial to total derivative wrt space, is intended to convey the idea that the second relation is seen as an ordinary differential equation in space, where the Laplace variable plays the role of a parameter. Then

$$U(x,p) = a(p)\,\cosh\big(\frac{p}{c}x\big) + b(p)\,\sinh\big(\frac{p}{c}x\big)\,. \tag{5}$$

The two unknown functions of p are defined by the two boundary conditions,

$$
\begin{aligned}
&(\mathrm{BC})_1 \quad U(x,p) = 0 \Rightarrow a(p) = 0 \\
&(\mathrm{BC})_2 \quad \frac{\partial U}{\partial x}(L,p) = \frac{F_0}{ES}\,\frac{1}{p} = b(p)\,\frac{p}{c}\,\cosh\,(\frac{p}{c}L) \\
&U(x,p) = c\,\frac{F_0}{ES}\,\frac{1}{p^2}\,\frac{\sinh\,(\frac{p}{c}x)}{\cosh\,(\frac{p}{c}L)}\,.
\end{aligned}
\tag{6}
$$

Let $T = L/c$ be the time for the longitudinal wave to travel the length of the bar. Then, the displacement of the right boundary $x = L$ is

$$
u(L,t) = \frac{F_0}{ES}\,c\,\mathcal{L}^{-1}\{\frac{1}{p^2}\,\tanh(T\,p)\}(t) = 2\,L\,\frac{F_0}{ES}\,f_3(t)\,,
\tag{7}
$$

where $f_3(t)$ is a periodic function shown on Fig. I.7. Use has been made of the previous exercise with $a = 2T$. Therefore the velocity of the boundary $x = L$ reads,

$$
\frac{\partial u}{\partial t}(L,t) = 2\,L\,\frac{F_0}{ES}\,\underbrace{\frac{df(t)}{dt}}_{\pm 1/2T} = \pm c\,\frac{F_0}{ES}\,.
\tag{8}
$$

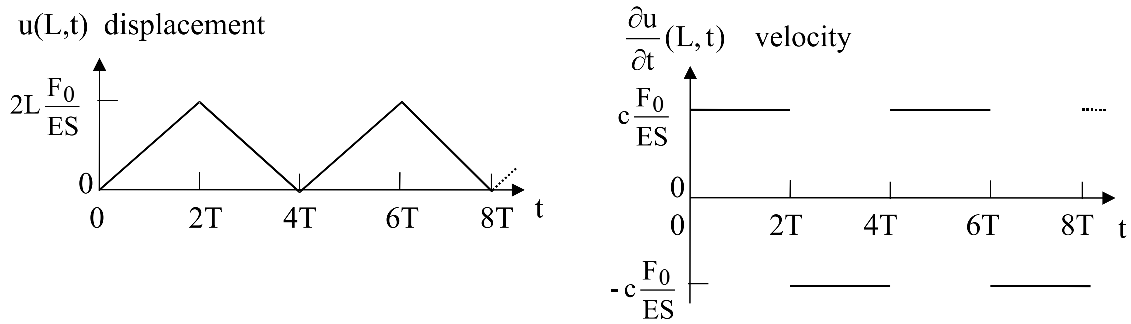

Figure I.7 Following the shock, the ensuing longitudinal wave propagates back and forth, giving rise to a periodic motion with a period $4T$ equal to four times the time required for the longitudinal elastic wave to travel the bar.

The interpretation of these longitudinal vibrations is a bit tricky and it goes as follows:

- at time $t = 0$, the right boundary of the bar is hit, and the corresponding signal propagates to the left at speed c;
- it needs a time T to reach the fixed boundary. When the wave comes back, it carries the information that this boundary is fixed;
- this information is known by the right boundary after a time $2T$. Since it was not known before, this point was moving to the right, with the positive velocity indicated by (8). Immediately as the information is known, the right boundary stops moving right, and in fact, changes the sign of its velocity, again as indicated by (8);
- the mechanical information “the right boundary is subject to a fixed traction” then travels back to the left, and hits the fixed boundary at time $3T$. When the wave comes back, it carries the information that this boundary is fixed. This information reaches the right boundary at time $4T$, where in fact its displacement just vanishes;

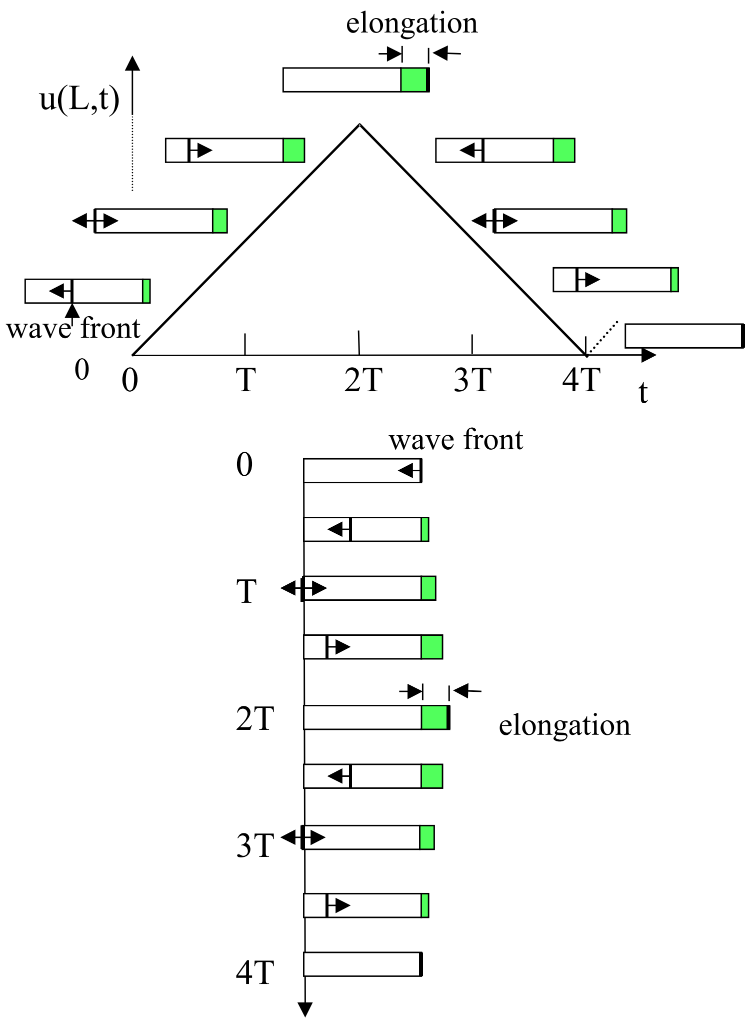

Figure I.8 Two other equivalent illustrations of the back and forth propagation of the wave front in the finite bar with left end fixed and right end subject to a given traction.

- the elongation increases linearly in time from $t = 0$ to reach its maximum at $t = 2T$, and then decreases up to $t = 4T$ where it vanishes;
- this succession of events, with periodicity $4T$, repeats indefinitely.

Figs. I.7 and I.8 display the position of the wave front and elongation at various times within a period.

Exercise I.4: **Transverse vibrations of a beam**.

A beam, of length L, is simply supported at its boundaries $x = 0$ and $x = L$. Consequently , the bending moments, linked to the curvature by the Navier-Bernouilli relation $M = EI\, d^2w/dx^2$, vanish at the boundaries. Here EI is the bending stiffness, and $w(x, t)$ the transverse displacement.

The beam is at rest for times $t < 0$. At time $t = 0$, it is submitted to a transverse shock expressed in terms of the transverse velocity $\partial w/\partial t(x, 0)$.

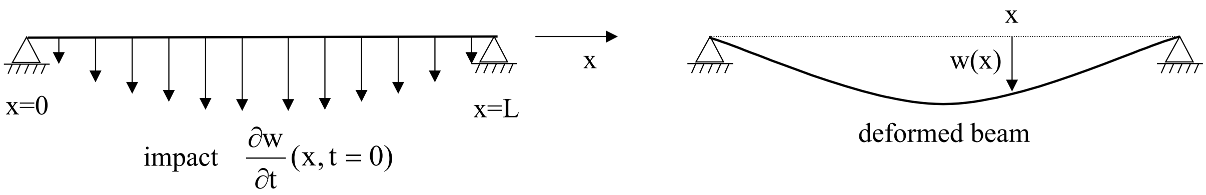

Figure I.9 The supports at the boundaries are bilateral, that is, they prevent up and down vertical motions of the beam.

The equations of dynamic linear elasticity governing the transverse displacement $w(x, t)$ of the beam are,

$$\begin{aligned}
&\text{(FE) field equation} \quad \frac{\partial^2 w}{\partial t^2} + b^2\,\frac{\partial^4 w}{\partial x^4} = 0, \quad t > 0,\; x \in]0, L[;\\
&\text{(IC) initial conditions} \quad w(x, 0) = 0; \quad \frac{\partial w}{\partial t}(x, 0) = V_0 \sin\left(\pi\,\frac{x}{L}\right);\\
&\text{(BC) boundary conditions} \quad w(0, t) = w(L, t) = 0; \quad \frac{\partial^2 w}{\partial x^2}(0, t) = \frac{\partial^2 w}{\partial x^2}(L, t) = 0\,.
\end{aligned} \tag{1}$$

The coefficient b involved in the field equation is equal to $\sqrt{EI/(\rho\, S)}$, where ρ is the mass density of the material and S the section of the beam, all quantities that we will consider as constants.

The beam remains uncharged. The transverse shock generates transverse vibrations $w(x, t)$. Describe these so-called free vibrations.

Solution:

The solution is obtained through the Laplace transform in time,

$$w(x, t) \to W(x, p) = \mathcal{L}\{w(x, t)\}(p), \tag{2}$$

and we admit that the operators transform and partial derivative in space Laplace commute,

$$\frac{\partial}{\partial x}\mathcal{L}\{w(x, t)\}(p) = \mathcal{L}\{\frac{\partial}{\partial x}w(x, t)\}(p)\,. \tag{3}$$

Therefore,

$$\begin{aligned}
\text{(FE)} \quad & b^2\,\frac{\partial^4 W(x, p)}{\partial x^4} + p^2\, W(x, p) - p\,\underbrace{w(x, 0)}_{=0,\ \text{(IC)}} - \underbrace{\frac{\partial w}{\partial t}(x, 0)}_{=V_0 \sin(\pi\, x/L),\ \text{(IC)}} = 0\\
& b^2\,\frac{d^4 W(x, p)}{dx^4} + p^2\, W(x, p) = V_0\,\sin(\pi\,\frac{x}{L})\,.
\end{aligned} \tag{4}$$

The change of notation, from partial to total derivative wrt space, is intended to convey the idea that the second relation is seen as an ordinary differential equation in space, where the Laplace variable plays the role of a parameter.

The solution of this nonhomogeneous linear differential equation is equal to the sum of the solution of the homogeneous equation (with zero rhs), and of a particular solution.

The solution of the homogeneous equation is sought in the format $W(w,p) = C(p)\,\exp(\alpha\,x)$, yielding four complex solutions $\alpha = \pm(1 \pm i)\,\beta$, with $\beta(p) = \sqrt{p/(2\,b)}$, and with an priori complex factor $C(p)$. Summing these four solutions, the real part may be rewritten in the format,

$$\begin{aligned} W^{\text{hom}}(x,p) &= \Big(c_1(p)\,\cos(\beta\,x) + c_2(p)\,\sin(\beta\,x)\Big)\,\exp(\beta\,x) \\ &+ \Big(c_3(p)\,\cos(\beta\,x) + c_4(p)\,\sin(\beta\,x)\Big)\,\exp(-\beta\,x)\,, \end{aligned} \tag{5}$$

where the $c_i = c_i(p)$, $i = 1-4$, are unknowns to be defined later.

The particular solution is sought in the form of the rhs,

$$W^{\text{par}}(x,p) = c_5(p)\,\sin(\pi\,\frac{x}{L}),\quad c_5(p) = \frac{V_0}{p^2 + b^2\,\dfrac{\pi^4}{L^4}}\,. \tag{6}$$

We will need the first two derivatives of the solution,

$$\begin{aligned} W(x,p) &= \Big(c_1(p)\,\cos(\beta\,x) + c_2(p)\,\sin(\beta\,x)\Big)\,\exp(\beta\,x) \\ &+ \Big(c_3(p)\,\cos(\beta\,x) + c_4(p)\,\sin(\beta\,x)\Big)\,\exp(-\beta\,x) \\ &+ c_5(p)\,\sin(\pi\,\frac{x}{L}) \\ \frac{d}{dx}W(x,p) &= \beta\,\Big((c_1(p)+c_2(p))\,\cos(\beta\,x) + (-c_1(p)+c_2(p))\,\sin(\beta\,x)\Big)\,\exp(\beta\,x) \\ &+ \beta\,\Big((-c_3(p)+c_4(p))\,\cos(\beta\,x) - (c_3(p)+c_4(p))\,\sin(\beta\,x)\Big)\,\exp(-\beta\,x) \\ &+ c_5(p)\,\frac{\pi}{L}\,\cos(\pi\,\frac{x}{L}) \\ \frac{d^2}{dx^2}W(x,p) &= 2\,\beta^2\,\Big(c_2(p)\,\cos(\beta\,x) - c_1(p)\,\sin(\beta\,x)\Big)\,\exp(\beta\,x) \\ &+ 2\,\beta^2\,\Big(-c_4(p)\,\cos(\beta\,x) + c_3(p)\,\sin(\beta\,x)\Big)\,\exp(-\beta\,x) \\ &- c_5(p)\,\frac{\pi^2}{L^2}\,\sin(\pi\,\frac{x}{L})\,, \end{aligned} \tag{7}$$

The four boundary conditions are used to obtain the four unknowns,

$$\begin{aligned} &c_1(p) + c_3(p) = 0 \\ &c_2(p) - c_4(p) = 0 \\ &\Big(c_1(p)\,\cos(\beta\,L) + c_2(p)\,\sin(\beta\,L)\Big)\,\exp(\beta\,L) + \Big(c_3(p)\,\cos(\beta\,L) + c_4(p)\,\sin(\beta\,L)\Big)\,\exp(-\beta\,L) = 0 \\ &(c_2(p)\,\cos(\beta\,L) - c_1(p)\,\sin(\beta\,L)\Big)\,\exp(\beta\,L) + (-c_4(p)\,\cos(\beta\,L) + c_3(p)\,\sin(\beta\,L)\Big)\,\exp(-\beta\,L) = 0\,, \end{aligned} \tag{8}$$

yielding $c_1 = -c_3$, $c_2 = c_4$, and a 2×2 linear system for c_1 and c_2,

$$\begin{aligned} \sinh(\beta L)\,\cos(\beta L) \quad c_1(p) \quad &+ \quad \cosh(\beta L)\,\sin(\beta L) \quad c_2(p) \quad = 0, \\ -\cosh(\beta L)\,\sin(\beta L) \quad c_1(p) \quad &+ \quad \sinh(\beta L)\,\cos(\beta L) \quad c_2(p) \quad = 0\,. \end{aligned} \tag{9}$$

The determinant of this system, $4\,(\sinh^2(\beta L) + \sin^2(\beta L))$, does not vanish, and therefore, $c_i = 0$, $i = 1-4$, and finally,

$$W(x,p) = W^{\rm par}(x,p) = \frac{V_0}{p^2 + b^2\,\dfrac{\pi^4}{L^4}}\,\sin(\pi\,\frac{x}{L})\,. \tag{10}$$

Since

$$\mathcal{L}\{\sin(a\,t)\}(p) = \frac{a}{p^2 + a^2}\,, \tag{11}$$

the transverse displacement, and transverse velocity,

$$w(x,t) = V_0\,\frac{L^2}{b\,\pi^2}\,\sin\Big(\frac{b\,\pi^2}{L^2}\,t\Big)\sin(\pi\,\frac{x}{L}), \quad \frac{\partial w}{\partial t}(x,t) = V_0\,\cos\Big(\frac{b\,\pi^2}{L^2}\,t\Big)\sin(\pi\,\frac{x}{L})\,, \tag{12}$$

are periodic with a frequency,

$$\frac{\pi}{2}\,\frac{b}{L^2}\,, \tag{13}$$

that is inversely proportional to the square of the length of the beam: doubling the length of the beam divides by four its frequency of vibration. On the other hand, the higher the transverse stiffness, the higher the frequency.

Since the solution displays a separation of the space and time variables, all points of the beam vibrate in phase, and the beam keeps its spatial shape for ever.

Exercise I.5: **A finite bar with ends at controlled temperature**.

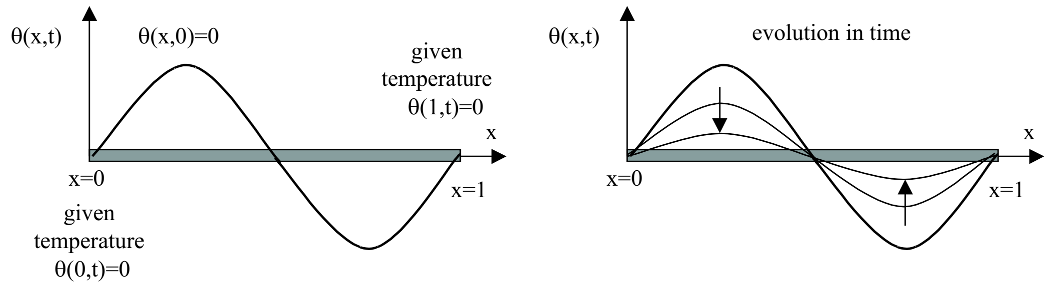

Figure I.10 Heat diffusion in a finite bar subject to given temperature at its ends.

The temperature at the two ends of a finite bar $x \in [0,1]$ is maintained at a given value, say $T(0,t) = T(1,t) = T_0$. The initial temperature along the bar $T(x,0)$=$T(x,0)$ is a function of space, say $T(x,0)$. It is more convenient to work with the field $\theta(x,t) = T(x,t) - T_0$ than with the temperature itself. Moreover, to simplify the notation, scaling of time and space has been used so as to make the thermal conductivity equal to one.

The initial and boundary value problem (IBVP) is thus governed by the following set of equations:

$$\begin{aligned}
&\text{field equation (FE)} \quad \frac{\partial^2 \theta}{\partial x^2} - \frac{\partial \theta}{\partial t} = 0, \quad t > 0, \ x \in]0,1[;\\
&\text{initial condition (IC)} \quad \theta(x,0) = \theta_0 \, \sin(2\,\pi\,x)\,;\\
&\text{boundary conditions (BC)} \quad \theta(0,t) = \theta(1,t) = 0\,.
\end{aligned} \tag{1}$$

Find the temperature $\theta(x,t)$ along the bar at time $t > 0$.

Solution:

The solution is obtained through the Laplace transform

$$\theta(x,t) \to \Theta(x,p) = \mathcal{L}\{\theta(x,t)\}(p)\,. \tag{2}$$

Therefore,

$$\text{(FE)} \ \frac{\partial^2 \Theta(x,p)}{\partial x^2} - \Big(p\,\Theta(x,p) - \underbrace{\theta(x,0)}_{=\theta_0\,\sin(2\,\pi\,x),\ \text{(IC)}}\Big) = 0\,. \tag{3}$$

The solution to this nonhomogeneous linear equation is the sum of the solution to the homogeneous equation, and of a particular solution. The latter is sought in the form of the inhomogeneity. Therefore,

$$\Theta^{\text{par}}(x,p) = c(p)\,\sin(2\,\pi\,x), \quad c(p) = \frac{\theta_0}{p + 4\,\pi^2}\,. \tag{4}$$

Therefore,

$$\Theta(x,p) = c_1(p)\,e^{\sqrt{p}\,x} + c_2(p)\,e^{-\sqrt{p}\,x} + \frac{\theta_0}{p + 4\,\pi^2}\,\sin(2\,\pi\,x)\,. \tag{5}$$

The boundary conditions imply the unknowns $c_1(p)$ and $c_2(p)$ to vanish. Therefore

$$\Theta(x,p) = \frac{\theta_0}{p + 4\,\pi^2}\,\sin\left(2\,\pi\,x\right), \tag{6}$$

and

$$\theta(x,t) = \theta_0\,\sin\left(2\,\pi\,x\right) e^{-4\,\pi^2\,t}\,\mathcal{H}(t)\,. \tag{7}$$

The spatial temperature profile remains identical in time, but its variation with respect to the temperature imposed at the boundaries decreases and ultimately vanishes. In other words, the information imposed at the boundaries penetrates progressively the body.

Since the temperature is imposed at the ends, the heat fluxes $\nabla\theta(x,t)$ at these ends $x = 0, 1$ can be seen as the response of the structure to a constraint.

Exercise I.6: **Relations around the complementary error function** erfc.

1. Show that

$$\mathcal{L}^{-1}(\exp(-a\sqrt{p})(t) = \frac{a}{2\sqrt{\pi\, t^3}}\exp(-\frac{a^2}{4\,t}), \quad a > 0\,, \tag{1}$$

by forming a differential equation.

2. Deduce, using the convolution theorem,

$$\mathcal{L}^{-1}\Big(\frac{1}{p}\exp(-a\sqrt{p})\Big)(t) = \mathrm{erfc}\Big(\frac{a}{2\sqrt{t}}\Big), \quad a > 0\,. \tag{2}$$

Proof:

1.1 The function $\sqrt{p}$ should be made uniform by defining appropriate cuts. One can then calculate the derivatives of $F(p) = \exp(-\sqrt{p})$,

$$F(p) = e^{-\sqrt{p}}, \quad \frac{d}{dp}F(p) = -\frac{e^{-\sqrt{p}}}{2\sqrt{p}}, \quad \frac{d^2}{dp^2}F(p) = \frac{e^{-\sqrt{p}}}{4\,p} + \frac{e^{-\sqrt{p}}}{4\,p\sqrt{p}}\,. \tag{3}$$

Therefore,

$$4\,p\,\frac{d^2}{dp^2}F(p) + 2\,\frac{d}{dp}F(p) - F(p) = 0\,. \tag{4}$$

If $f(t)$ has Laplace transform $F(p)$, let us recall the rules,

$$\begin{aligned}
&\mathcal{L}\{t\,f(t)\}(p) = -\frac{d}{dp}F(p),\\
&\mathcal{L}\{t^2 f(t)\}(p) = (-1)^2\frac{d^2}{dp^2}F(p),\\
&\mathcal{L}\{\frac{d}{dt}\Big(t^2 f(t)\Big)\}(p) = p\,\mathcal{L}\{t^2 f(t)\}(p) - \underbrace{\Big(t^2 f(t)\Big)(t=0)}_{=0} = p\,\frac{d^2}{dp^2}F(p)\,.
\end{aligned} \tag{5}$$

That the second term on the rhs of the last line above is really zero need to be checked, once the solution has been obtained. Collecting these relations, the differential equation in the Laplace domain (4) can be transformed into a differential equation in time,

$$4\,\frac{d}{dt}\Big(t^2 f(t)\Big) - 2\,t\,f(t) - f(t) = 0\,, \tag{6}$$

which, upon expansion of the first term, becomes,

$$\frac{df}{f} + \Big(\frac{3}{2}\frac{1}{t} - \frac{1}{4}\frac{1}{t^2}\Big)\,dt = 0\,, \tag{7}$$

and thus can be integrated to,

$$f(t) = \frac{c}{t^{3/2}}e^{-1/(4t)}\,. \tag{8}$$

1.2 The constant c can be obtained as follows. We will need a generalized Abel theorem, that we can state as follows:

Assume that, for large t, the two functions $f(t)$ and $g(t)$ are sufficiently close, $f(t) \simeq g(t)$, for $t \gg 1$. Then their Laplace transforms $t \to p$ are also close for small p, namely $F(p) \simeq G(p)$, for $p \sim 0$.

Consider now,

$$t\,f(t)=\frac{c}{t^{1/2}}e^{-1/(4t)},\quad \mathcal{L}\{t\,f(t)\}(p)=-\frac{d}{dp}F(p)=\frac{e^{-\sqrt{p}}}{2\,\sqrt{p}}\,. \tag{9}$$

Then

$$\begin{aligned}&\text{for } t\gg 1,\quad t\,f(t)\simeq\frac{c}{t^{1/2}}\quad\Rightarrow\quad \mathcal{L}\{t\,f(t)\}(p)\simeq c\,\frac{\Gamma(1/2)}{p^{1/2}},\\ &\text{for } p\sim 0,\quad \frac{e^{-\sqrt{p}}}{2\,\sqrt{p}}\simeq\frac{1}{2\,\sqrt{p}}\,.\end{aligned} \tag{10}$$

Requiring the transforms in these two lines to be equal as indicated by (9), and by the generalized Abel theorem, yields $c\,\Gamma(1/2)=1/2$, and therefore $c=1/(2\sqrt{\pi})$. Therefore

$$\mathcal{L}^{-1}(\exp(-\sqrt{p})(t)=\frac{1}{2\sqrt{\pi\,t^3}}\exp(-\frac{1}{4\,t})\,. \tag{11}$$

The rule, for real $\alpha\geq 0$,

$$\mathcal{L}\{t^\alpha\,\mathcal{H}(t)\}(p)=\frac{\Gamma(\alpha+1)}{p^{\alpha+1}}\,, \tag{12}$$

involves the tabulated function Γ, with the properties $\Gamma(n+1)=n!$ for $n\geq 0$ integer and $\Gamma(1/2)=\sqrt{\pi}$.

1.3 Using the rule

$$\mathcal{L}\{f(b\,t)\}(p)=\frac{1}{b}\,\mathcal{L}\{f(t)\}(\frac{p}{b})\,, \tag{13}$$

with $b>0$ a constant, the result (11) may be generalized to (1), setting $b=1/a^2$. $\square$

2. Indeed, using the convolution theorem and (1),

$$\begin{aligned}\mathcal{L}^{-1}\Big(\frac{1}{p}\exp(-\sqrt{p}\,a)\Big)(t) &= \int_0^t \mathcal{H}(t-u)\,\frac{a}{2\sqrt{\pi\,u^3}}\exp(-\frac{a^2}{4\,u})\,du\\ &= \frac{2}{\sqrt{\pi}}\int_{a/(2\sqrt{t})}^\infty e^{-v^2}\,dv,\quad\text{with}\quad v^2=a^2/(4\,u)\,.\end{aligned} \tag{14}$$

$\square$

Exercise I.7: **a multipurpose contour integration**.

The purpose is the inversion of the one-sided Laplace transform $Q(x,p)$,

$$Q(x,p) = \frac{\exp(-x\sqrt{p/a})}{p-b}, \tag{1}$$

where $a > 0$, $b \geq 0$ are constants, $x \geq 0$ is the space coordinate and p the Laplace variable associated with time t. Show that the inverse reads,

$$q(x,t) = \frac{\exp(b\,t)}{2}\Big(\exp\big(x\sqrt{\frac{b}{a}}\big)\,\mathrm{erfc}\big(\frac{x}{2\sqrt{at}} + \sqrt{b\,t}\big) + \exp\big(-x\sqrt{\frac{b}{a}}\big)\,\mathrm{erfc}\big(\frac{x}{2\sqrt{at}} - \sqrt{b\,t}\big)\Big). \tag{2}$$

Proof:

The function is first made uniform by introducing a branch cut along the negative axis $\Re p \leq 0$, and the definitions,

$$p = |p|\,\exp(i\,\theta), \quad \theta \in]-\pi,\pi], \quad \sqrt{p} = \sqrt{|p|}\,\exp(i\,\theta/2) \quad (\Rightarrow \Re\sqrt{p} \geq 0). \tag{3}$$

Assuming known the (inverse) transforms,

$$\mathcal{L}^{-1}\{\exp(-x\sqrt{\frac{p}{a}})\}(u) = \frac{x}{2\sqrt{\pi\,a\,u^3}}\exp(-\frac{x^2}{4\,a\,u}), \quad \mathcal{L}^{-1}\{\frac{1}{p-b}\}(u) = \exp(b\,u)\,\mathcal{H}(u), \tag{4}$$

the function $q(x,t)$ can be written

$$q(x,t) = \exp(b\,t)\,\frac{2}{\sqrt{\pi}}\int_{x/2\sqrt{a\,t}}^{\infty} \exp(-v^2 - \frac{b\,x^2}{4\,a\,v^2})\,dv, \tag{5}$$

since the Laplace transform of a convolution product is equal to the product of the Laplace transforms.

This integral seems out of reach, except if $b = 0$, where

$$q(x,t) = \exp(b\,t)\,\mathrm{erfc}(\frac{x}{2\sqrt{a\,t}}). \tag{6}$$

Still it is provided in explicit form in Abramowitz and Stegun [1964], p. 304, for any b.

Since we want to obtain the result on our own, even for $b \neq 0$, we take a step backward and consider the inversion in the complex plane through an appropriate closed contour $C = C(R,\epsilon)$, that respects the branch cut and includes the pole $p = b$. A preliminary finite contour is shown on Fig. I.11. The residue theorem yields,

$$\begin{aligned} &\frac{1}{2\,i\,\pi}\Big(\int_{c-i\,R}^{c+i\,R} + \int_{C_R} + \int_{C_\epsilon} + \int_{C^+} + \int_{C^-}\Big)\ \exp(tp)\,Q(x,p)\,dp \\ &= \frac{1}{2\,i\,\pi}\oint_C \exp(t\,p)\,Q(x,p)\,dp = \exp(b\,t - x\sqrt{b/a}), \end{aligned} \tag{7}$$

where c is an arbitrary real, which is required to be greater than b for a correct definition of the inverse Laplace transform. For vanishingly small radius ϵ, the contour C_ϵ does not contribute as long as $b \neq 0$. If $b = 0$, then the inverse Laplace transfrm is read directly from (5). Moreover,

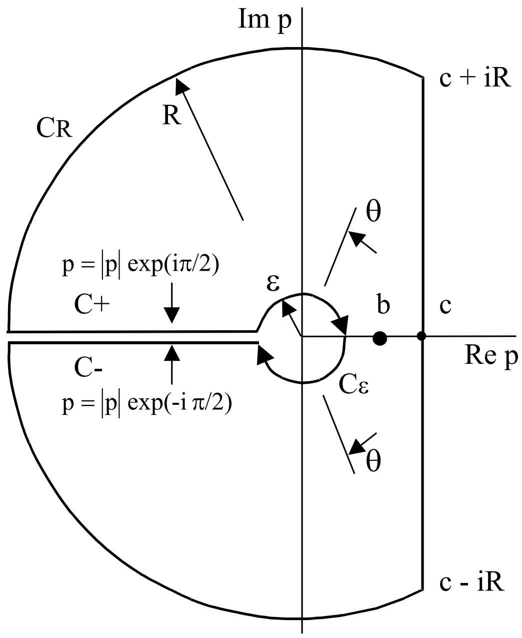

Figure I.11 Contour of integration associated with the integral (7).

since $Q(x,p)$ tends to 0 for large p in view of (3), the contour C_R does not contribute either for large R, in view of Jordan's lemma. Consequently,

$$\begin{aligned} q(x,t) &= \lim_{R\to\infty} \frac{1}{2\,i\,\pi} \int_{c-i\,R}^{c+i\,R} \exp(t\,p)\, Q(x,p)\, dp \\ &= \exp(b\,t - x\,\sqrt{b/a}) - \lim_{R\to\infty,\ \epsilon\to 0} \frac{1}{2\,i\,\pi} \Big(\int_{C+} + \int_{C-} \Big) \exp(t\,p)\, Q(x,p)\, dp\,. \end{aligned} \tag{8}$$

On the branch cut, $p = -r < 0$, but $\sqrt{p}$ is equal to $i\,\sqrt{r}$ on the upper part and to $-i\,\sqrt{r}$ on the lower part. Therefore, the integrals along the branch cut become,

$$\begin{aligned} I &= \frac{1}{2\,i\,\pi} \int_{\infty}^{0} \exp(-t\,r)\, \frac{\exp(-i\,x\,\sqrt{r/a})}{-r-b}\,(-dr) + \frac{1}{2\,i\,\pi} \int_{0}^{\infty} \exp(-t\,r)\, \frac{\exp(i\,x\,\sqrt{r/a})}{-r-b}\,(-dr) \\ &= \frac{1}{2\,i\,\pi} \int_{0}^{\infty} \frac{\exp(-t\,r)}{r+b}\,(\exp(i\,x\,\sqrt{r/a}) - \exp(-i\,x\,\sqrt{r/a}))\, dr \\ &= \frac{1}{i\,\pi} \int_{0}^{\infty} \frac{\exp(-t\,\rho^2)}{\rho^2+b}\,(\exp(i\,x\,\rho/\sqrt{a}) - \exp(-i\,x\,\rho/\sqrt{a}))\,\rho\, d\rho \\ &= \frac{1}{i\,\pi} \int_{-\infty}^{\infty} \exp\big(-t\,\rho^2 + i\,x\,\rho/\sqrt{a}\big)\, \frac{\rho}{\rho^2+b}\, d\rho\,. \end{aligned} \tag{9}$$

In an attempt to see the error function emerging, we make a change of variable that transforms, to within a constant, the argument of the exponential into a square, namely

$$\rho \longrightarrow v = \sqrt{t}\rho - i\,v_0, \quad v_0 \equiv \frac{x}{2\sqrt{at}}\,. \tag{10}$$

Then

$$I = \exp(-(\frac{x}{2\sqrt{at}})^2) \times \frac{1}{2\,i\,\pi} \int_{-\infty-i\,v_0}^{\infty-i\,v_0} \frac{\exp(-v^2)}{v+i\,v^+} + \frac{\exp(-v^2)}{v+i\,v^-}\, dv, \quad v^{\pm} \equiv v_0 \pm \sqrt{b\,t}\,. \tag{11}$$

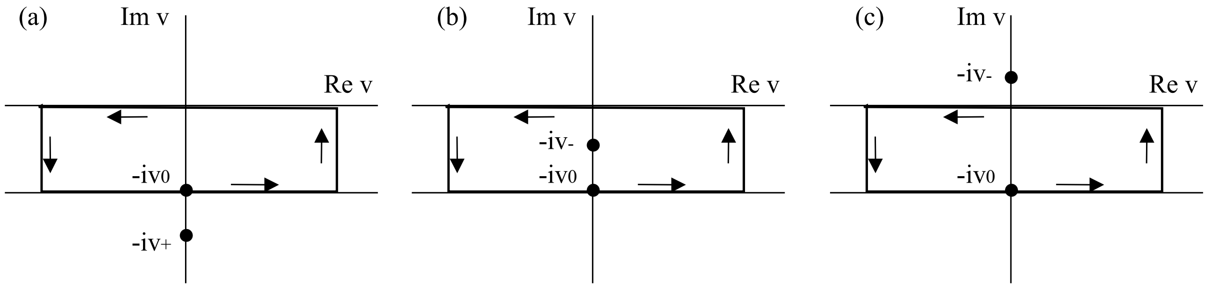

Figure I.12 Contours of integration associated with the integrals (12). Note that (a) v^+ is always larger than $v_0 > 0$, while v^- is (b) either between 0 and v^0 or (c) negative.

First observe that, by an appropriate choice of integration path, Fig. I.12, and application of the residue theorem,

$$\frac{1}{2\,i\,\pi}\int_{-\infty-i\,v_0}^{\infty-i\,v_0}\frac{\exp(-v^2)}{v+i\,v^\pm}\,dv=\frac{1}{2\,i\,\pi}\int_{-\infty}^{\infty}\frac{\exp(-v^2)}{v+i\,v^\pm}\,dv+\begin{cases}0 & \text{for } v^\pm=v^+ \text{ or } v^-<0,\\ \exp((v^-)^2) & \text{for } v^\pm=v^->0,\end{cases}\tag{12}$$

we are left with integrals on the real line. The basic idea is to insert the following identity,

$$\int_0^\infty \exp(-u\,X^2)\,du=\frac{1}{X^2},\quad X\neq 0\,,\tag{13}$$

in the integrals to be estimated and to exchange the order of integration. With this idea in mind, the following straightforward transformations are performed,

$$\begin{aligned}
&\frac{1}{2\,i\,\pi}\int_{-\infty}^{\infty}\frac{\exp(-v^2)}{v+i\,v^\pm}\,dv\\
=\;&-\frac{v^\pm}{\pi}\int_0^\infty\frac{\exp(-v^2)}{v^2+(v^\pm)^2}\,dv\quad(\text{use (13) })\\
=\;&-\frac{v^\pm}{\pi}\int_0^\infty\exp\big(-v^2-u\,(v^2+(v^\pm)^2)\big)\,du\,dv,\quad(du\,dv\to dv\,du,\quad v\to v\,\sqrt{1+u})\qquad(14)\\
=\;&-\frac{v^\pm}{2\,\sqrt{\pi}}\int_0^\infty\frac{\exp(-u\,(v^\pm)^2)}{\sqrt{1+u}}\,du\quad(u\to(\sinh v)^2,\quad |v^\pm|\cosh v\to w)\\
=\;&-\frac{\mathrm{sgn}(v^\pm)}{2}\exp\big((v^\pm)^2\big)\,\frac{2}{\sqrt{\pi}}\int_{|v^\pm|}^\infty\exp(-w^2)\,dw\,,
\end{aligned}$$

and therefore

$$\frac{1}{2\,i\,\pi}\int_{-\infty}^{\infty}\frac{\exp(-v^2)}{v+i\,v^\pm}\,dv=-\frac{\mathrm{sgn}(v^\pm)}{2}\exp\big((v^\pm)^2\big)\,\mathrm{erfc}(|v^\pm|)\,.\tag{15}$$

Collecting the results (12) and (15), and using the property $(\text{I.3.17})_3$,

$$\frac{1}{2\,i\,\pi}\int_{-\infty-i\,v_0}^{\infty-i\,v_0}\frac{\exp(-v^2)}{v+i\,v^\pm}\,dv=\begin{cases}-\frac{1}{2}\exp\big((v^+)^2\big)\,\mathrm{erfc}(v^+) & \text{for } v^\pm=v^+,\\ -\frac{1}{2}\exp\big((v^-)^2\big)\,\mathrm{erfc}(v^-)+\exp\big((v^-)^2\big) & \text{for } v^\pm=v^-,\end{cases}\tag{16}$$

Finally, the inversion formula deduces from (8) and (16).