Institut de Recherche en
Génie Civil et Mécanique
UMR CNRS 6183

UNIVERSITÉ DE NANTES

Centrale
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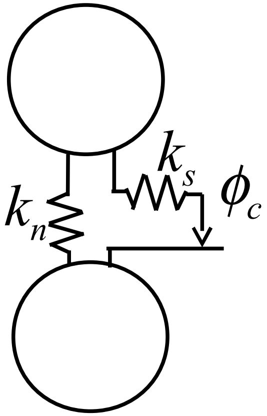
Directional stress probes to exhibit constitutive behaviour of discrete element models

Olek Zienkiewicz Course 2011
Discrete Mechanics of Geomaterials

Luc Sibille ***GeM Laboratory – Nantes University – ECN – CNRS***

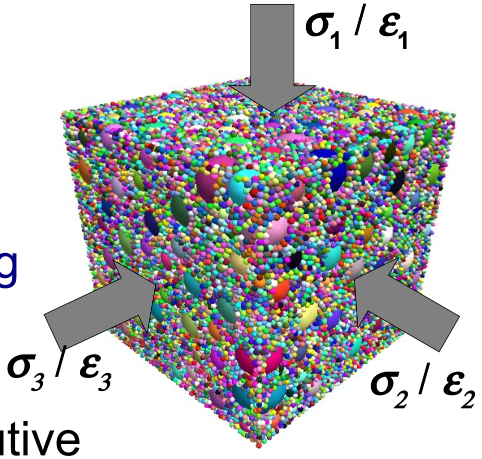
I. Introduction

I-1 Stress and strain probes, why?



- In DEM, physics is described at the contact and grain scale (contact stiffness, friction, adhesion, grain shape ...).

- At the macroscopic scale (over a REV) the mechanical behaviour (often complex) results from the collective response of particles (involving fabric, forces chains, force cycles, etc ...).



⇒ At the present time, there is no way to fully characterize *a priori* the macroscopic behaviour of the numerical granular assembly.

⇒ This situation is somehow similar to that of real (natural) granular matter.

⇒ The macroscopic mechanical behaviour can be exhibited through loading experiments.

- Gudehus (1979) proposed to characterize graphically incremental constitutive relations by plotting the response envelops to unit strain probes.

⇒ This approach based on response envelops can be extended to real materials (although it is technically painful, *Royis & Doanh 1998*), or to virtual (numerical) materials (much easier!)

I. Introduction

I-1 Rate independent material

- For rate independent materials: *(See Darve 1987-90)*

$$d\tilde{\varepsilon} = G_h(d\tilde{\sigma})$$

where G_h depends on the previous stress-strain history through the memory parameters h.

- $\forall \lambda \quad G_h(\lambda\, d\tilde{\sigma}) = \lambda\, G_h(d\tilde{\sigma})$ → G is homogeneous of degree 1 → Application of Euler's identity gives:

$$d\tilde{\varepsilon} = \frac{\partial G_h}{\partial(d\tilde{\sigma})} d\tilde{\sigma}$$

- Identifying $M_h(d\tilde{\sigma}) = \dfrac{\partial G_h}{\partial(d\tilde{\sigma})}$, M_h is homogeneous of degree 0 ($M_h(\lambda\, d\tilde{\sigma}) = M_h(d\tilde{\sigma})$)

→ M_h depends only on the direction $\tilde{u} = d\tilde{\sigma}/\|d\tilde{\sigma}\|$ of $d\tilde{\sigma}$ and not its norm.

$$d\tilde{\varepsilon} = M_h(\tilde{u})\, d\tilde{\sigma}$$

For $d\tilde{\varepsilon}$, $d\tilde{\sigma}$ expressed as pseudo vectors (in principal stress and strain direction for simplicity)

$$d\vec{\varepsilon} = M_h\left(\frac{d\vec{\sigma}}{\|d\vec{\sigma}\|}\right) d\vec{\sigma} \qquad d\vec{\varepsilon} = \begin{cases} d\varepsilon_1 \\ d\varepsilon_2 \\ d\varepsilon_3 \end{cases} \qquad d\vec{\sigma} = \begin{cases} d\sigma_1 \\ d\sigma_2 \\ d\sigma_3 \end{cases}$$

⇒ The incremental constitutive relation can be exhibited through strain (stress) responses to stress (strain) increments describing the different stress (strain) space directions.

I. Introduction

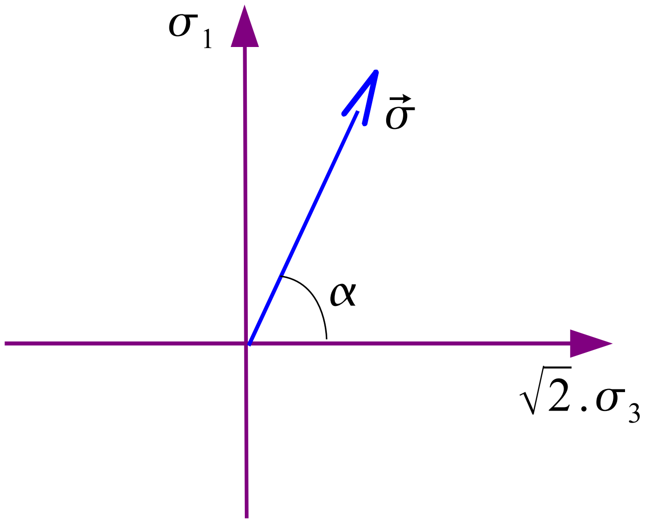
I-2 Rendulic stress or strain plane

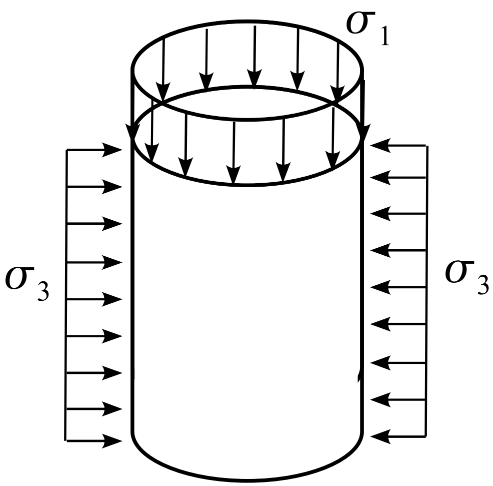
General restrictions:

- For the sake of simplicity we consider:
  - irrotational strains and stresses,
  - axisymmetric strain and stress states around axis `1' ($\sigma_2 = \sigma_3$ and $\varepsilon_2 = \varepsilon_3$),

where axes 1, 2 and 3 correspond to principal strain and stress directions (classical triaxial states)

⇒ A stress state is completely represented in the Rendulic stress plane (or axisymmetric stress plane).



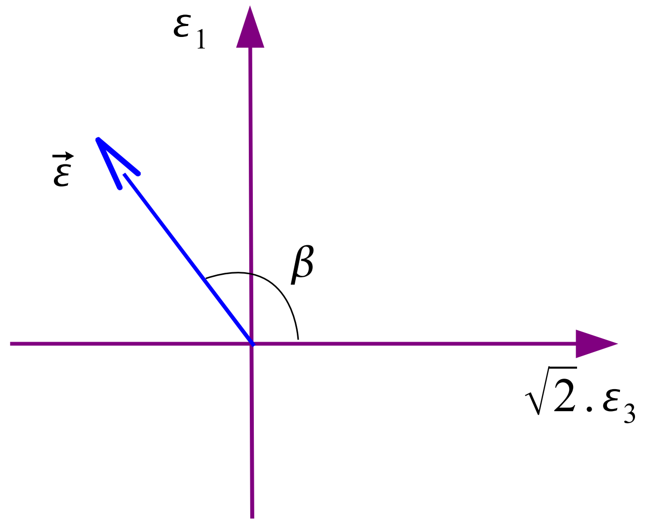


$$\|\vec{\sigma}\|=\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2}=\sqrt{\sigma_1^2+(\sqrt{2}.\sigma_3)^2}$$

I. Introduction

I-2 Rendulic stress or strain plane

⇒ A strain state is completely represented in the Rendulic strain plane (or axisymmetric strain plane).



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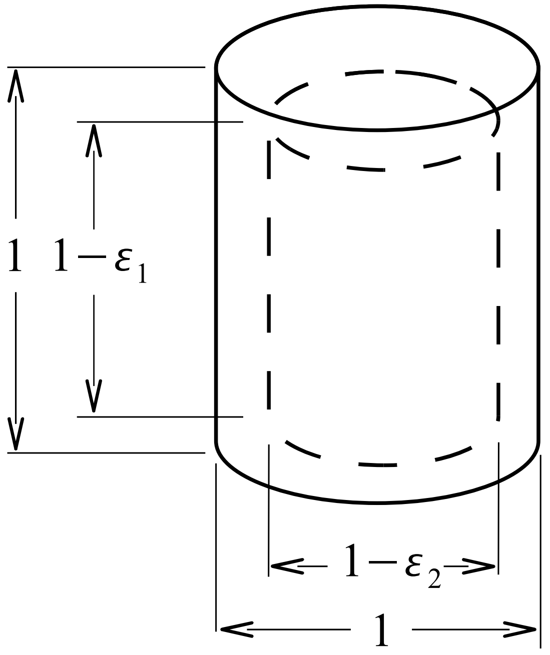
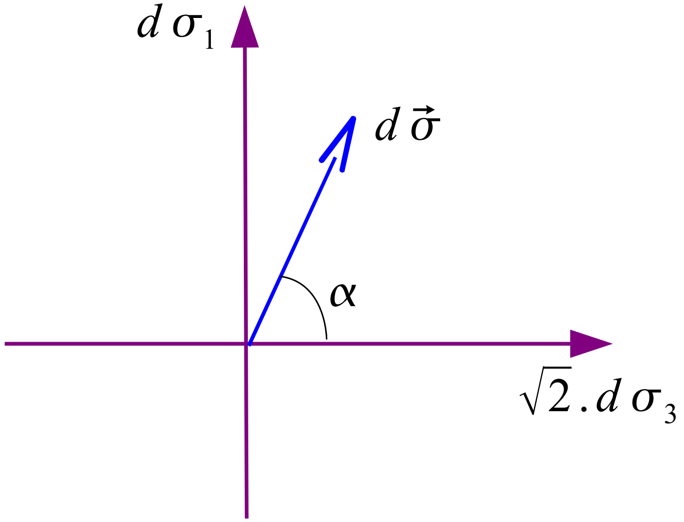


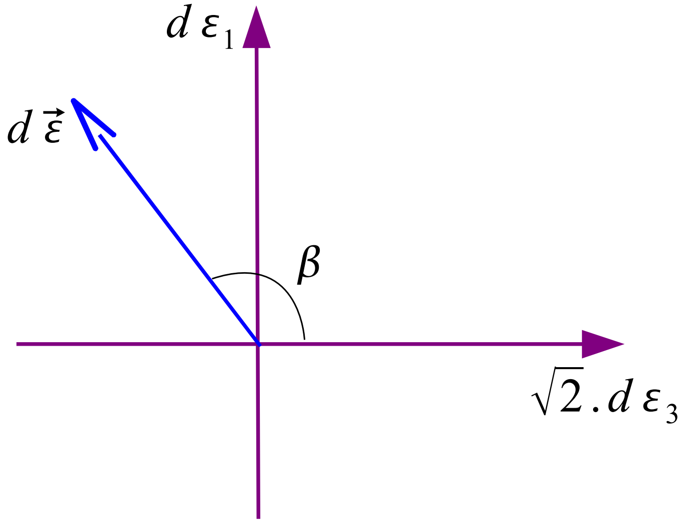
II. Directional stress/strain probes, and incremental response

Graphical characterization of material behaviour: $d\vec{\varepsilon} = M_h(\vec{u})\,d\vec{\sigma}$

- The material constitutive behaviour is represented by the incremental strain (stress) responses to “unit” stress increments applied in different directions.

⇒ Sequential loading of the material from the same initial state by a “unit” stress loading in different space directions (stress probes).





Same $\|d\vec{\sigma}\|$ but different α

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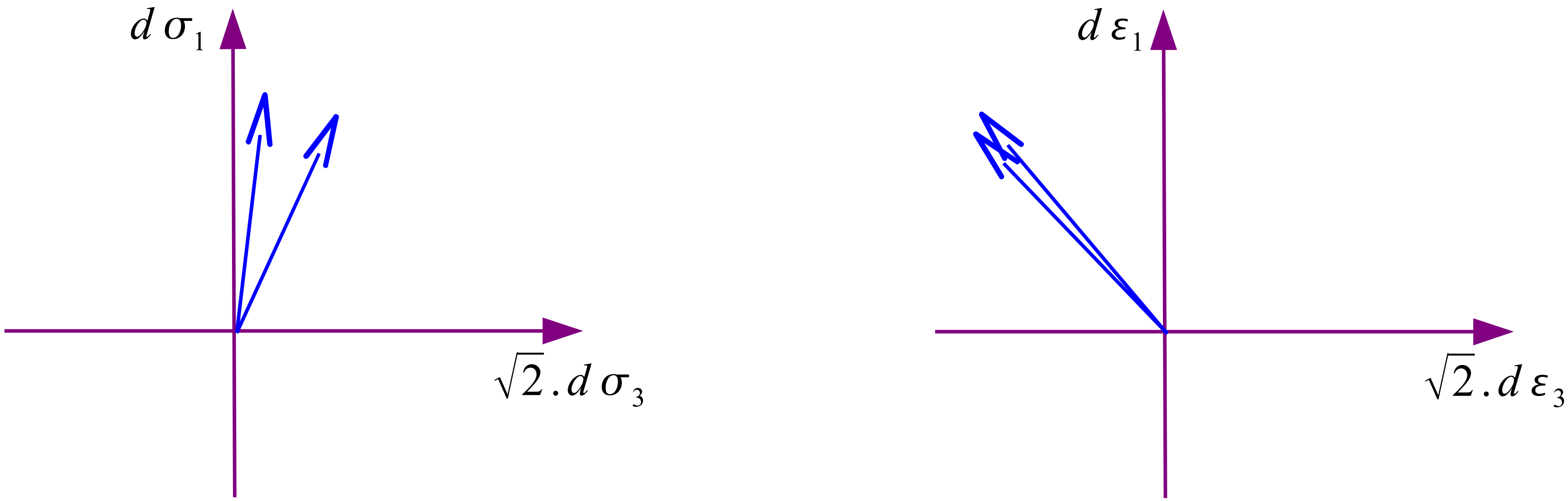

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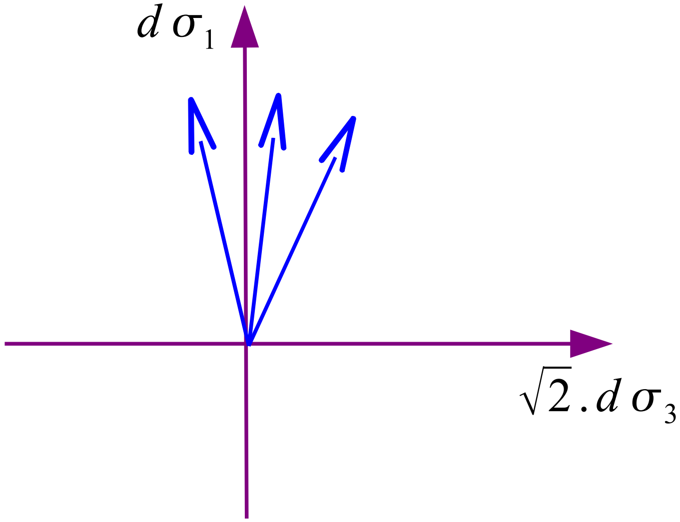

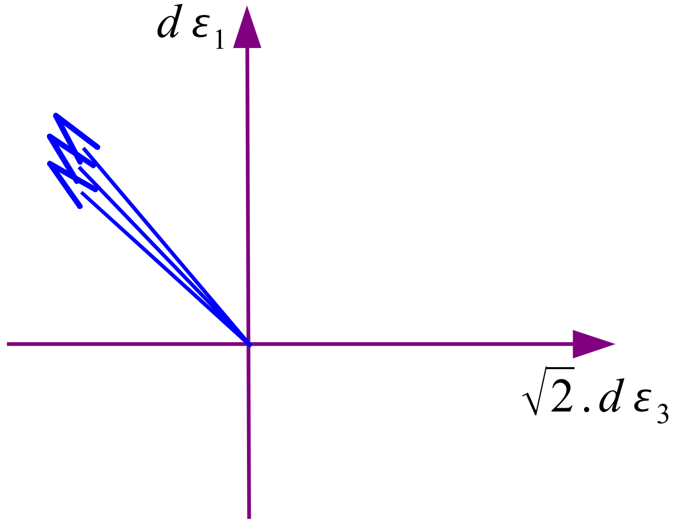

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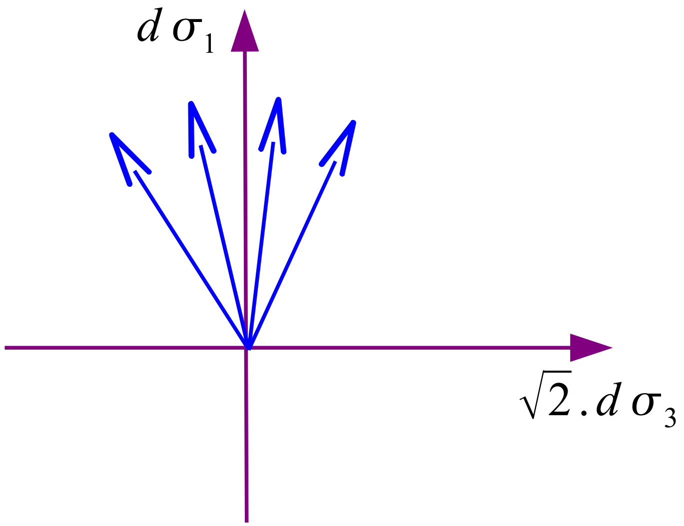

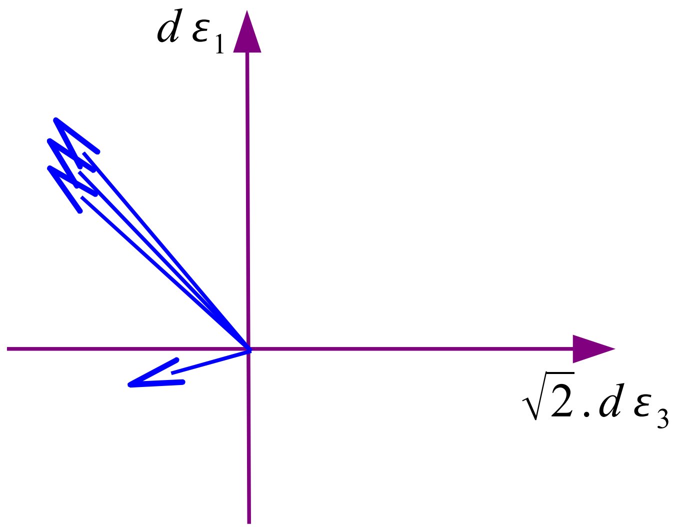

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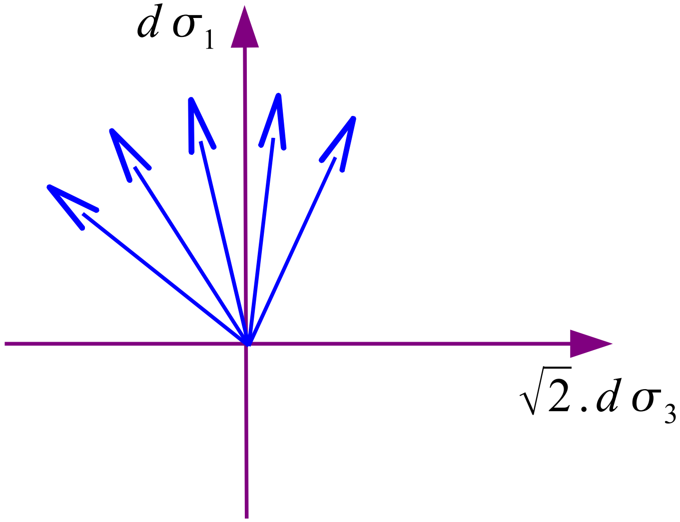

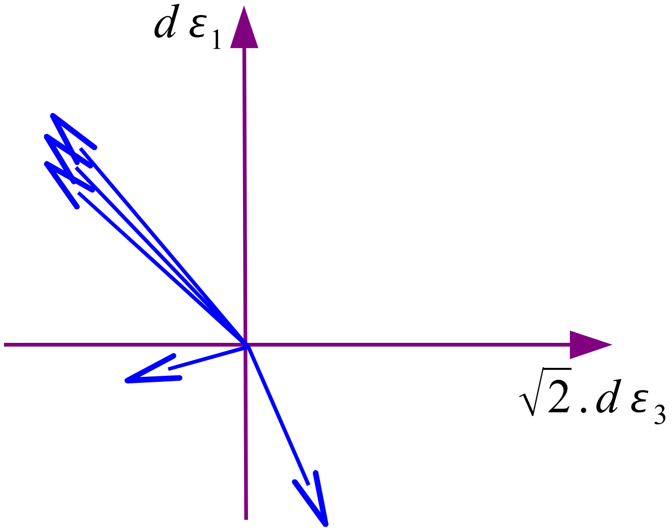

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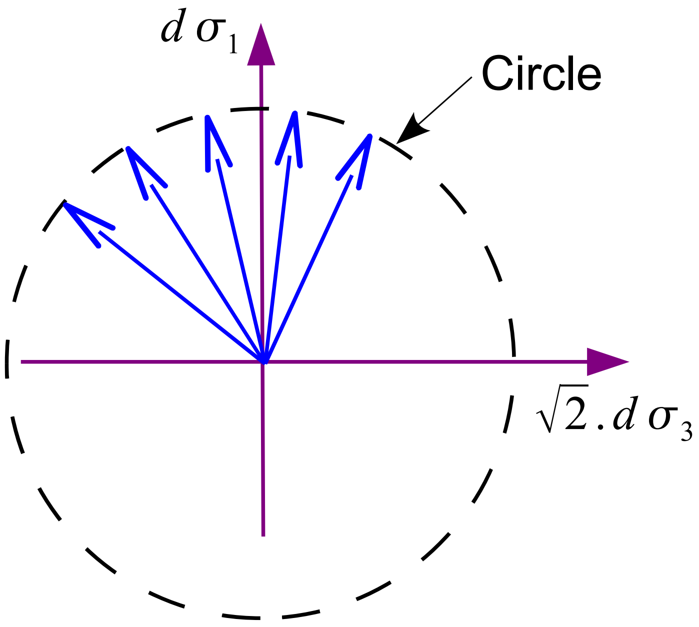
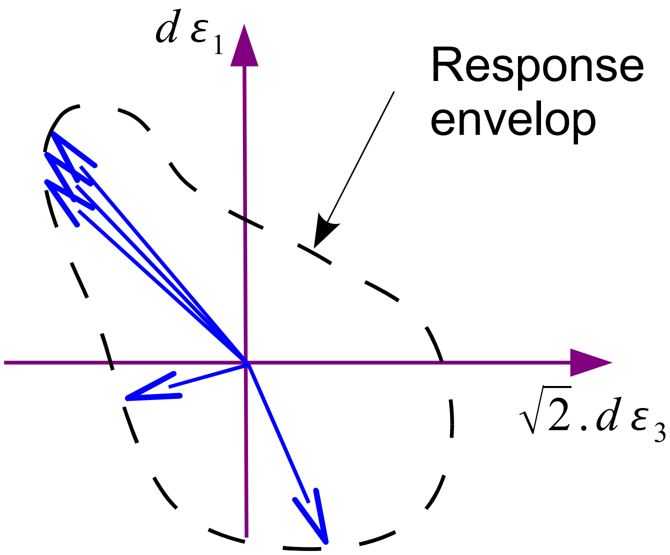
II. Directional stress/strain probes, and incremental response

Graphical characterization of material behaviour:





The envelop of the incremental strain (stress) responses is called the Gudehus strain (stress) response envelop.

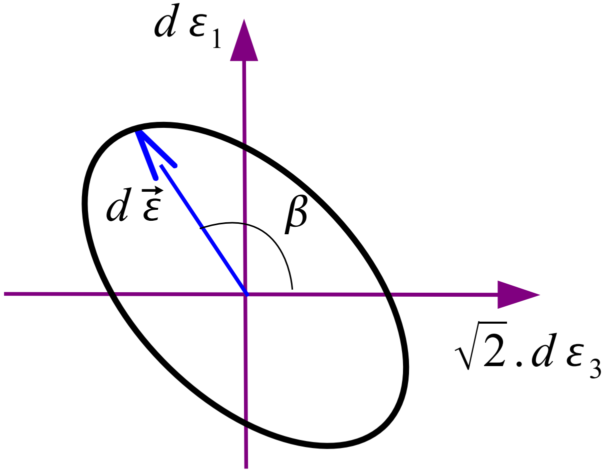
⇒ This representation is suitable to characterize the mechanical behaviour of rate-independent materials where the material response depends on the previous stress-strain history and loading direction.

⇒ The shape of the response envelop fully characterizes the constitutive behaviour

III. Typical response envelops

III-1 Incrementally linear response

Incrementally linear constitutive behaviour: $d\vec{\varepsilon} = M_h(\cancel{\vec{u}})\, d\vec{\sigma}$
a single linear relation between the strain increment and the stress increment, for all the stress loading directions (or strain directions) (i.e. there is a single tensorial zone)



⇒ The strain response envelop is an ellipse centred at the origin of the Rendulic strain increment plane.

(see details in: Gudehus 1979, Proc. 3rd Numer. Meth. in Geomechanics ; and Bardet 1994, Int. J. Plasticity)

Exemple: isotropic Hooke's law
(in principal stress and strain axes)

$$d\varepsilon_1 = \frac{1}{E}\left[d\sigma_1 - \nu(d\sigma_2 + d\sigma_3)\right]$$

$$d\varepsilon_2 = \frac{1}{E}\left[d\sigma_2 - \nu(d\sigma_1 + d\sigma_3)\right]$$

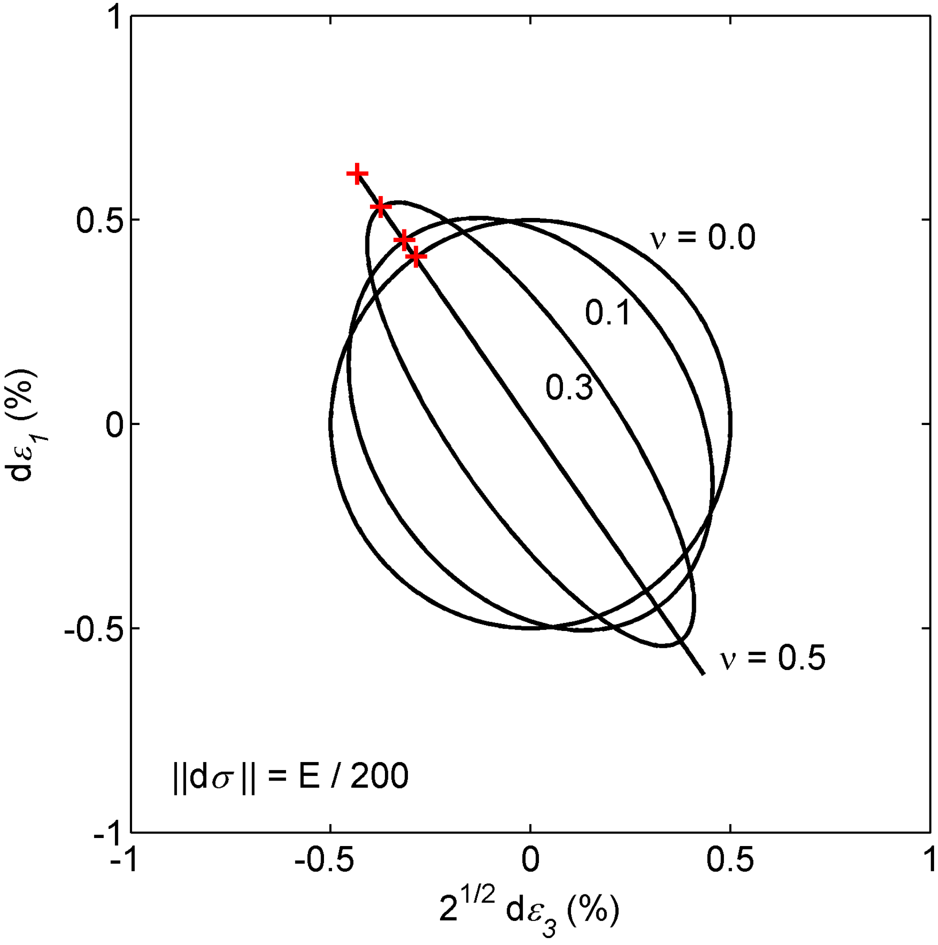
$$d\varepsilon_3 = \frac{1}{E}\left[d\sigma_3 - \nu(d\sigma_1 + d\sigma_2)\right]$$

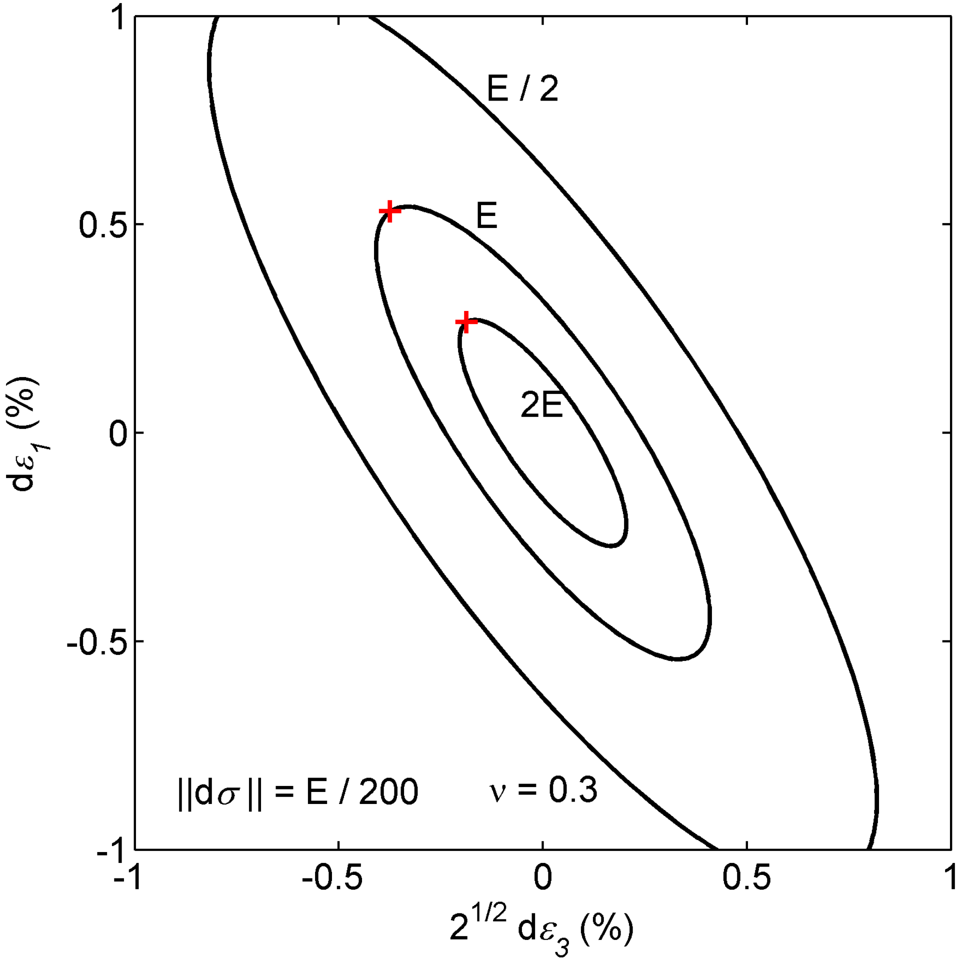
⇒ The strain response envelop is an ellipse. The size and the shape depend on E and ν only.

III. Typical response envelops

III-1 Incrementally linear response

Exemple: isotropic Hooke's law





⇒ shape of ellipse depends on ν only.

⇒ size of ellipse depends on E only.

III. Typical response envelops

III-1 Incrementally linear response

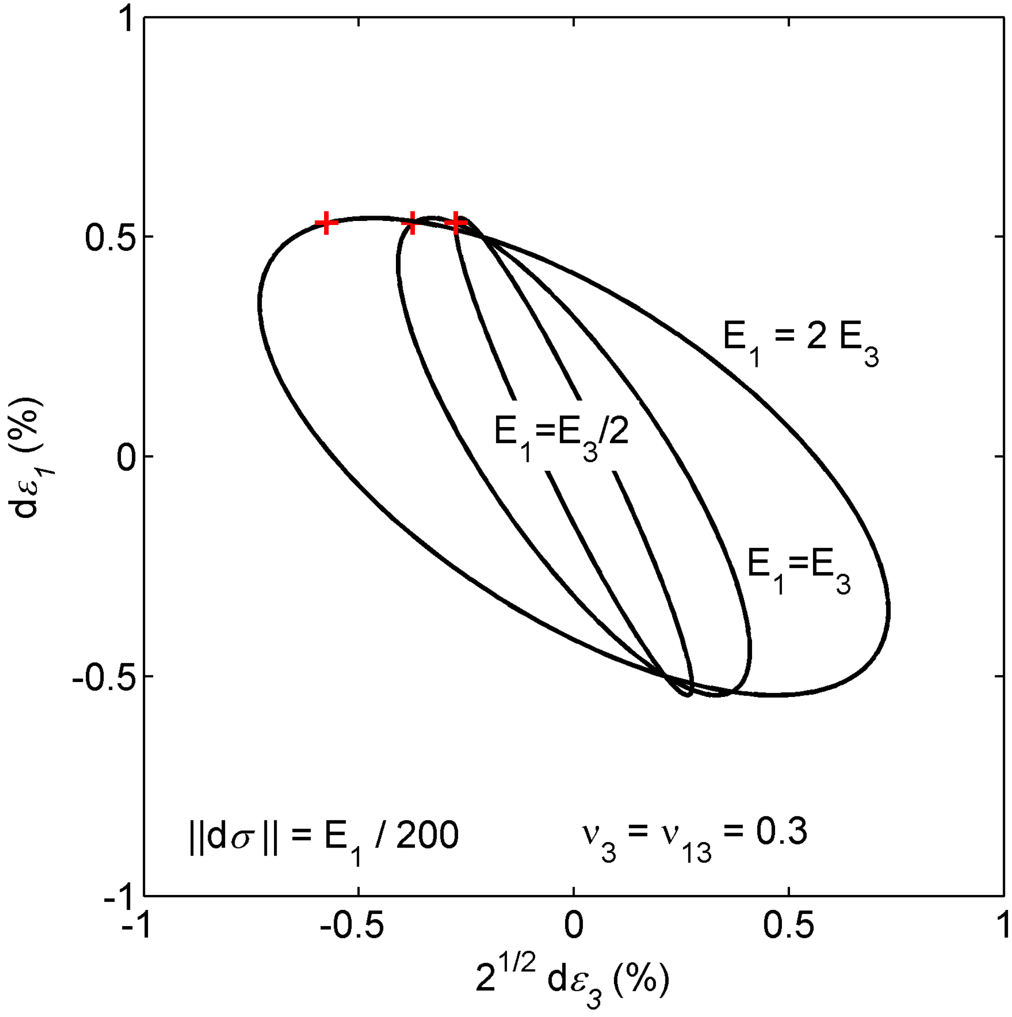
Anisotropic linear elasticity: case of the transverse isotropic Hooke's law

Same properties in directions 2 and 3:

$$E_2 = E_3 \qquad \nu_{23} = \nu_{32} = \nu_3$$

But different properties in direction 1:

$$E_1 \neq E_3 \qquad \nu_{13} \neq \nu_{31}$$



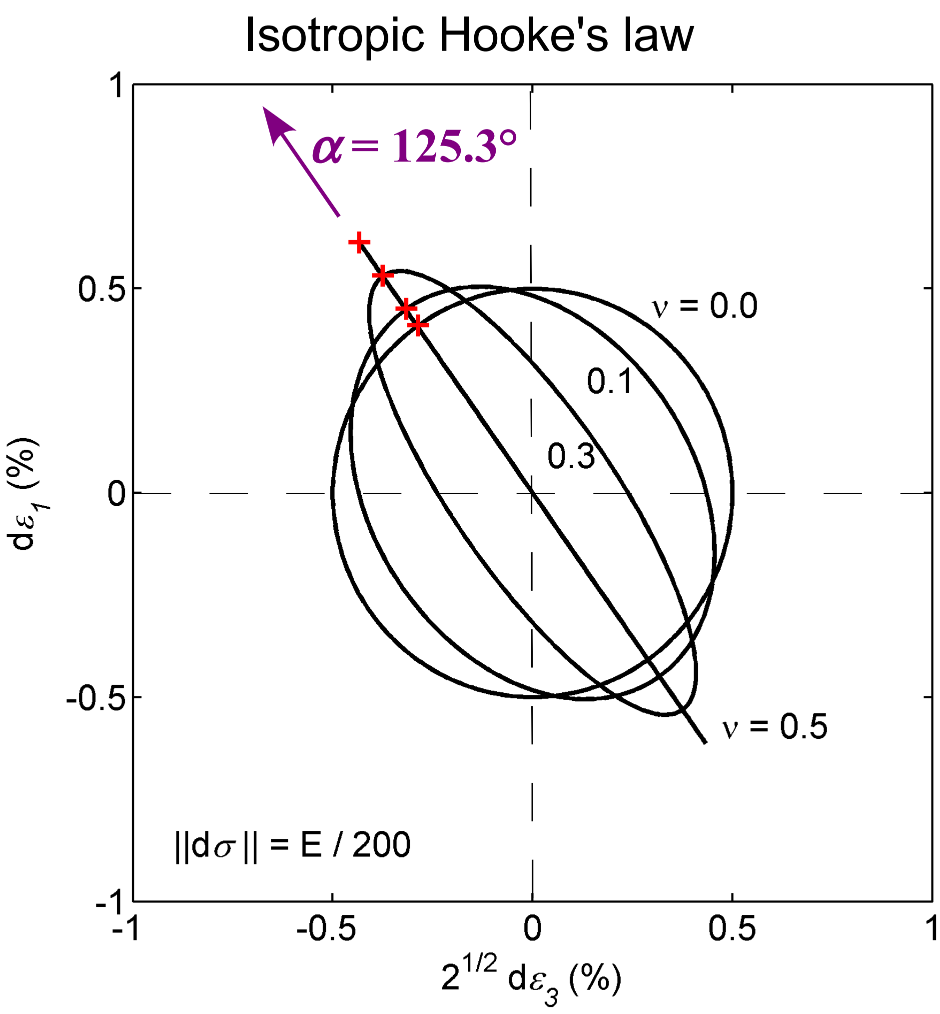
$$d\varepsilon_1 = \frac{d\sigma_1}{E_1} - 2\frac{\nu_{31}}{E_3} d\sigma_3$$

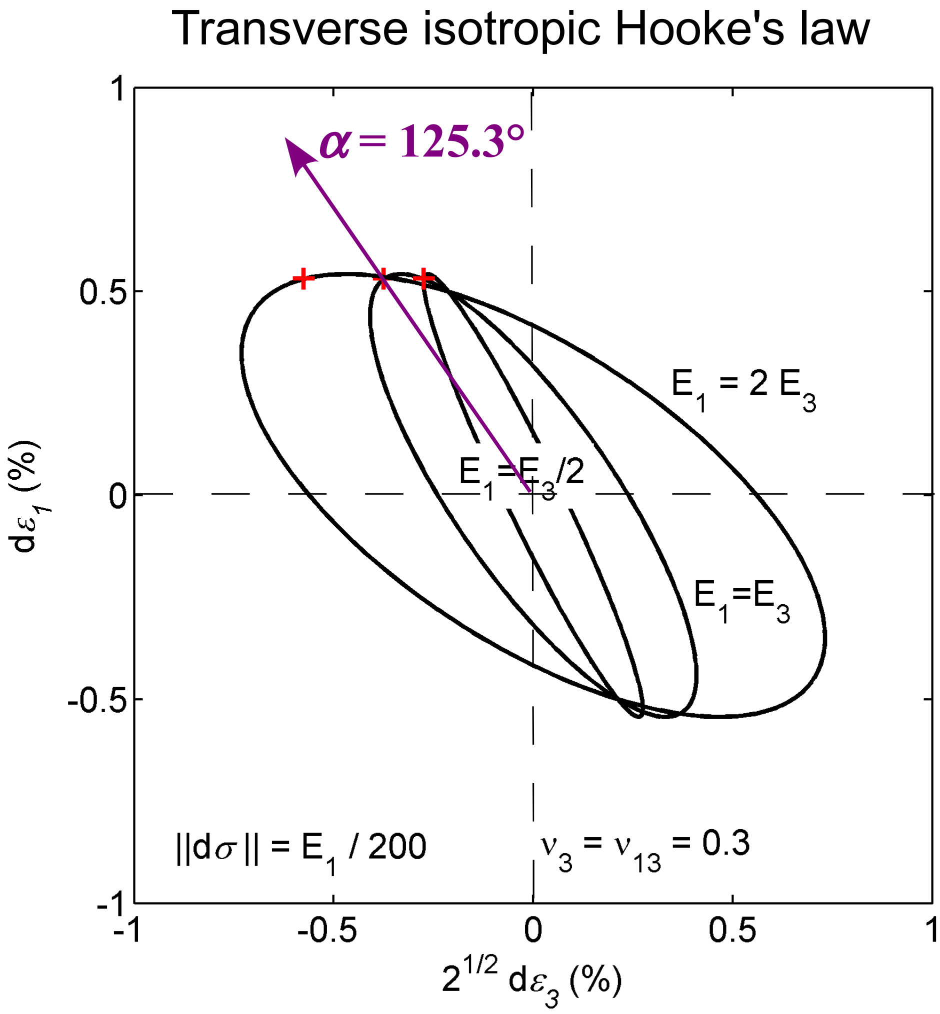
$$d\varepsilon_3 = -\frac{\nu_{13}}{E_1} d\sigma_1 - \frac{\nu_3}{E_3} d\sigma_3 + \frac{d\sigma_3}{E_3}$$

with: $\dfrac{\nu_{13}}{E_1} = \dfrac{\nu_{31}}{E_3}$

III. Typical response envelops

III-1 Incrementally linear response





Purely deviatoric stress increase: $dp = d\sigma_1 + 2\, d\sigma_3 = 0$ and $dq = d\sigma_1 - d\sigma_3 > 0 \;\Rightarrow\; \alpha = 125{,}3°$

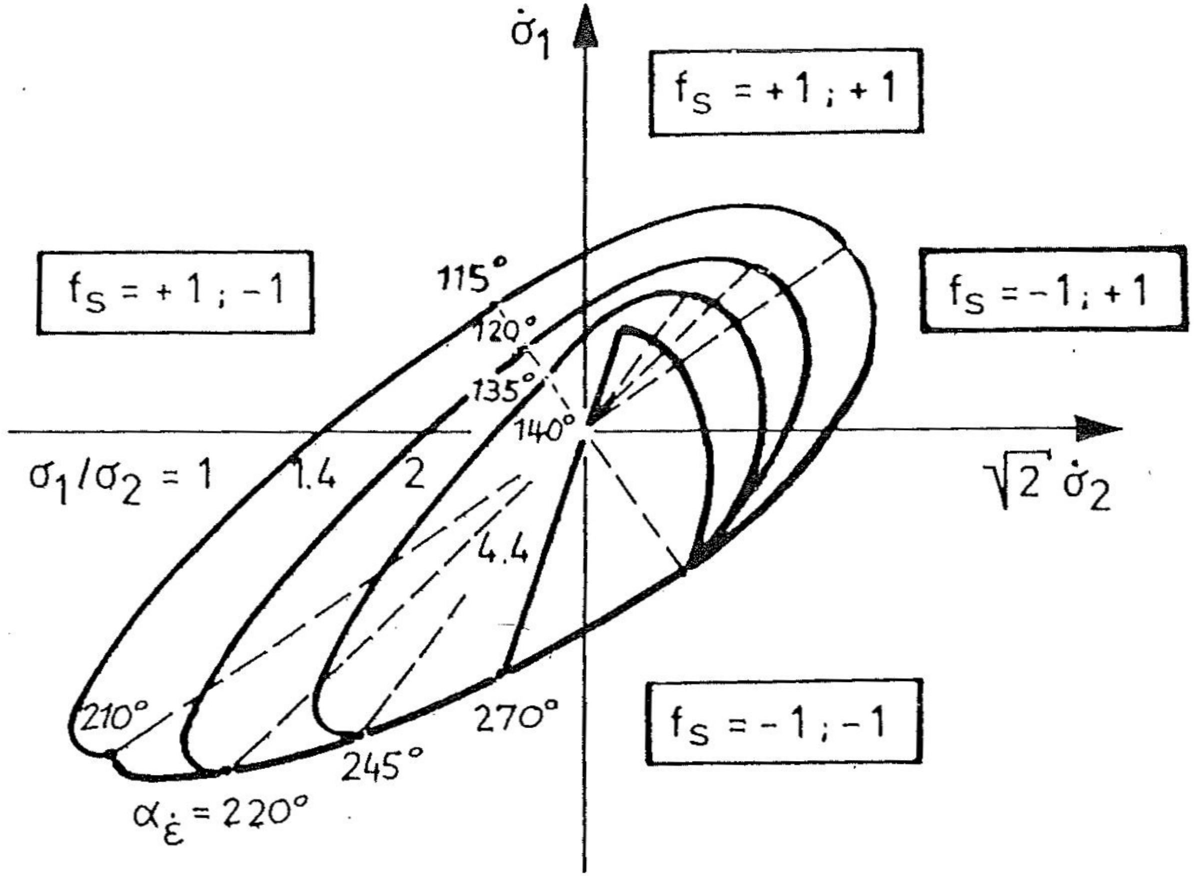
Isotropy ⇒ inclination of ellipse major axis: 125.3°, and direction of strain response to purely deviatoric stress loading aligned with ellipse major axis.

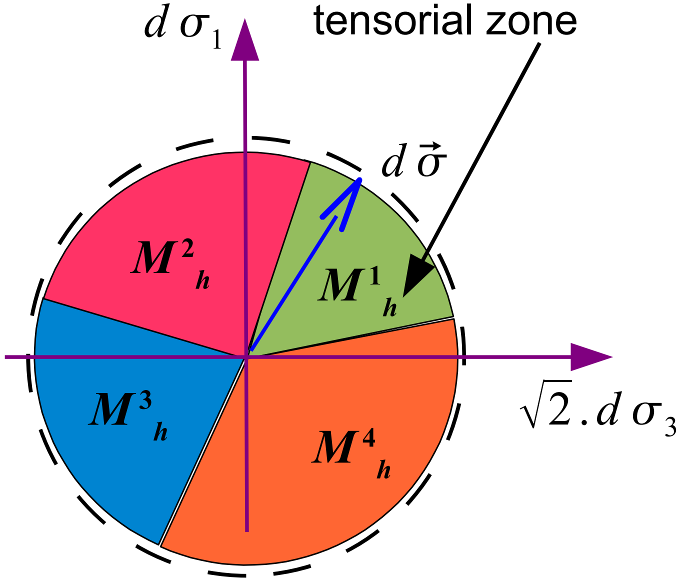
III. Typical response envelops

III-2 Incrementally piece-wise linear response

Exemple: elastoplastic law with two yield functions

(incrementally piece-wise linear relation with 4 tensorial zones)





Linear constitutive relation for a given tensorial zone

⇒ 4 pieces of ellipses centred at the origin of the Rendulic stress rate plane

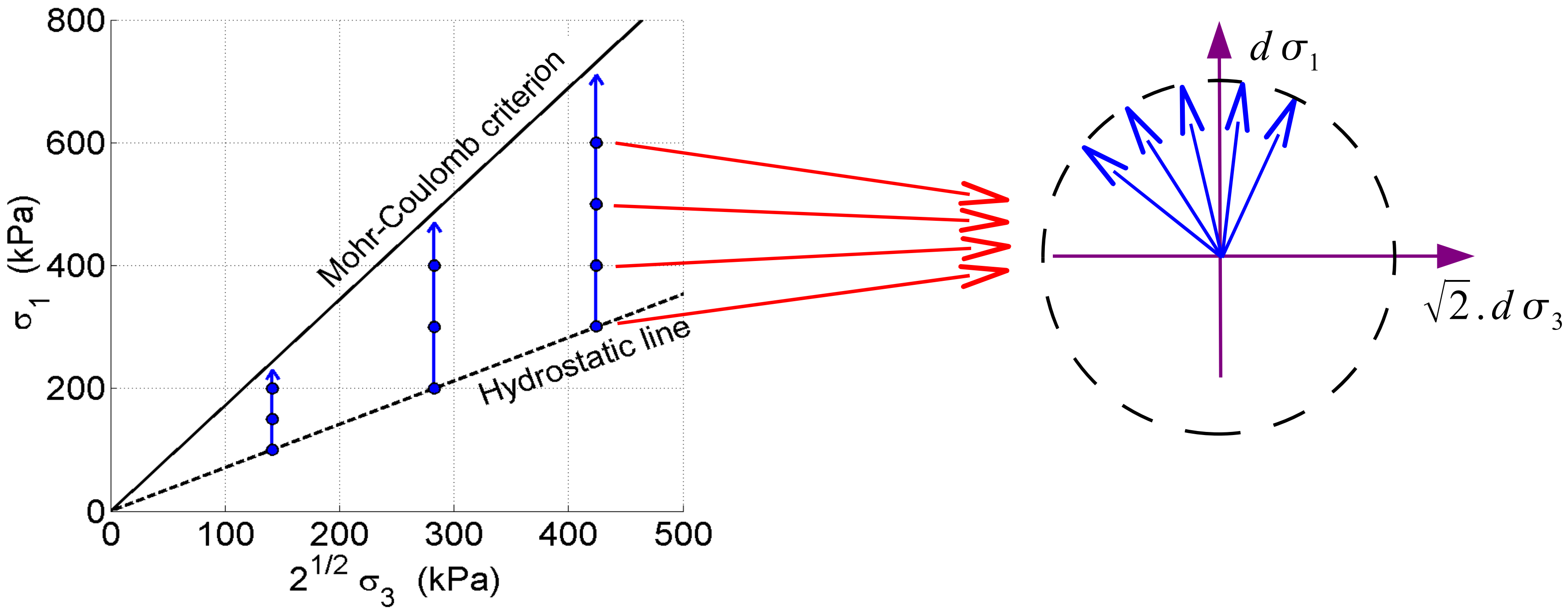
Gudehus G., " A comparison of some constitutive laws for soils under radially symmetric loading and unloading", Proc. 3rd Numer. Meth. in Geomechanics, A. A. Balkema, Aachen, p. 1309-1323, 2-6 April, 1979.

II. How to perform stress or strain probes with DEM

II-1 The loading programme

Case of stress probes

1. Reach an initial stress state and stabilize the granular assembly at this stress state.
2. Choose a value for the size of the increment of the stress loading $\|d\,\vec{\sigma}\|$.
3. Apply this stress increment in a given direction α from the initial stress state considered.
4. From the same initial stress state apply the same stress increment in a 2nd, 3rd, ... direction α.



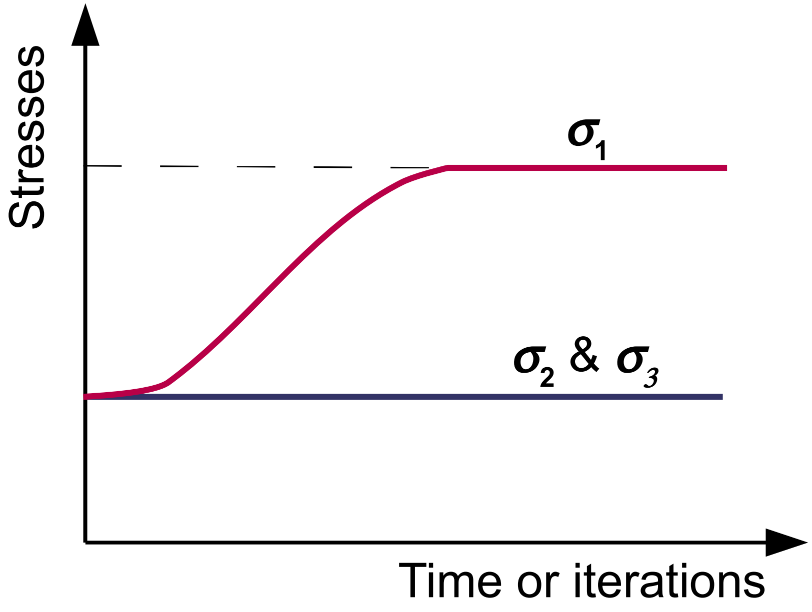
Classically, the initial stress states can be reached after an isotropic compression followed by a triaxial drained compression.

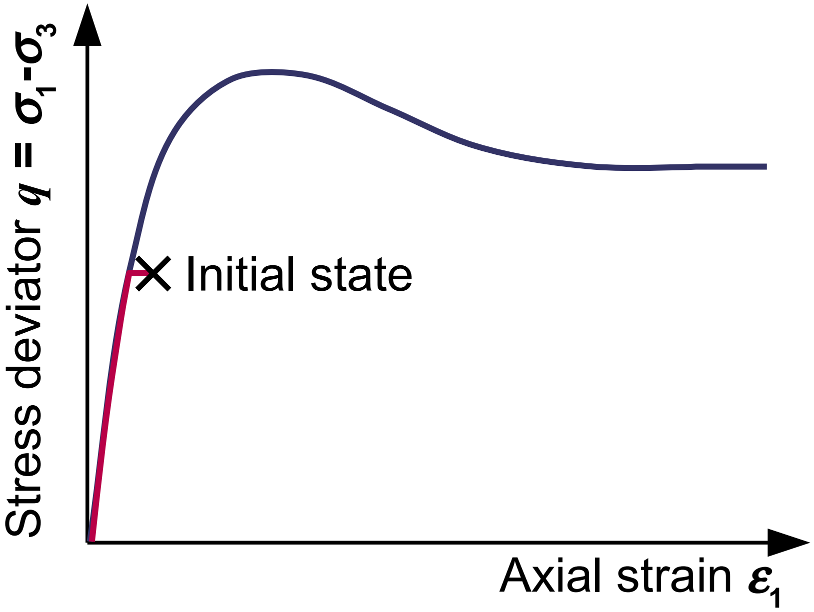
IV. How to perform stress or strain probes with DEM

IV-1 Initial stress state

Perform an triaxial drained compression with a full stress control

- The compression, even if it's stress controlled should be made slow enough to stay in a quasi-static strain regime.
- Stop the simulation when the quasi-equilibrium threshold is reached (with respect to the kinetic energy or the global unbalance force for instance).





Use for instance with Yade the ThreeDTriaxialEngine:

```
ThreeDTriaxialEngine(stressControl_1=1,  stressControl_2=1, stressControl_3=1,
                     sigma1=150e3, sigma2=100e3, sigma3=100e3,
                     strainRate1=0.05, strainRate2=100, strainRate3=100)
```

IV. How to perform stress or strain probes with DEM

IV-2 Stress loading increment

For rate-independant materials:

The mechanical response depends only on the history and the loading direction.

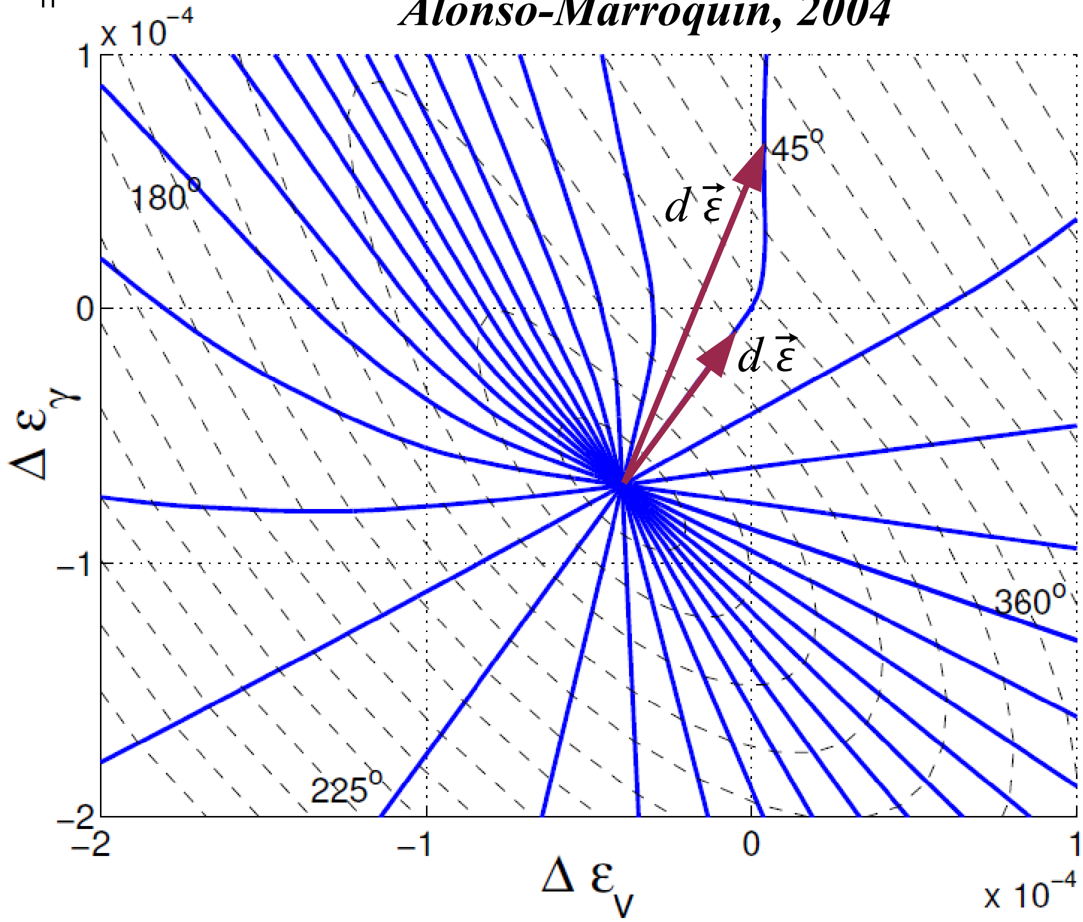
$$d\vec{\varepsilon} = M_h(\vec{u})\, d\vec{\sigma}$$

⇒ for stress probes:

- **the size of the of the strain response $\|d\vec{\varepsilon}\|$ should be proportional to the size of the stress increment $\|d\vec{\sigma}\|$,**

- **the direction of the strain response should be independent of the size of the stress increment $\|d\vec{\sigma}\|$.**

Alonso-Marroquin, 2004



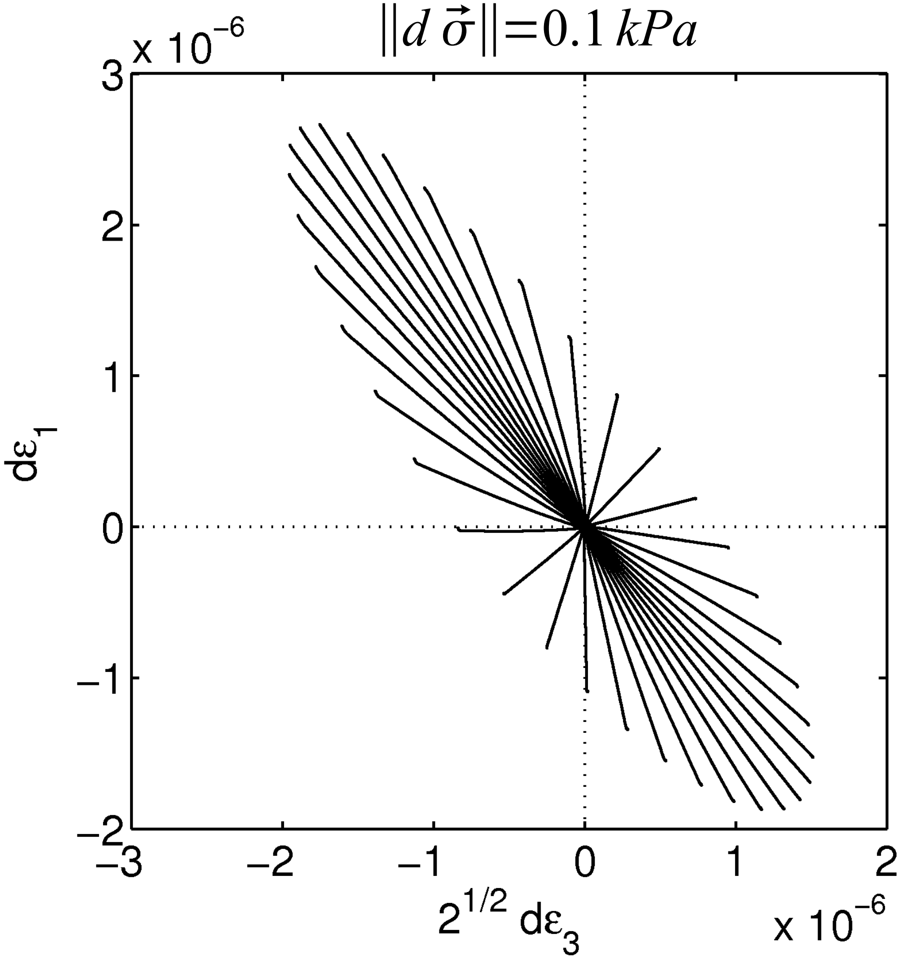
Practically not true: the strain response path to a stress loading applied in a given direction is not rectilinear.

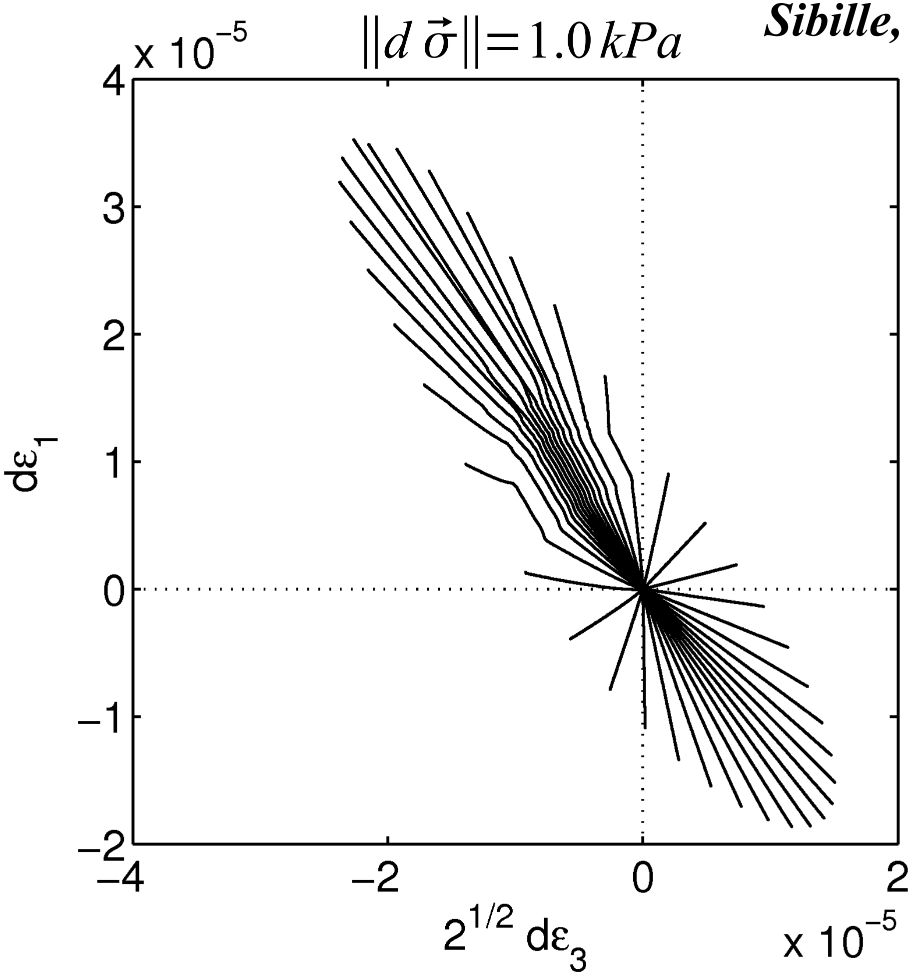
⇒ history run continuously

⇒ mechanisms at the origin of irreversible strain can be more or less discontinuous in time

IV. How to perform stress or strain probes with DEM

IV-2 Stress loading increment



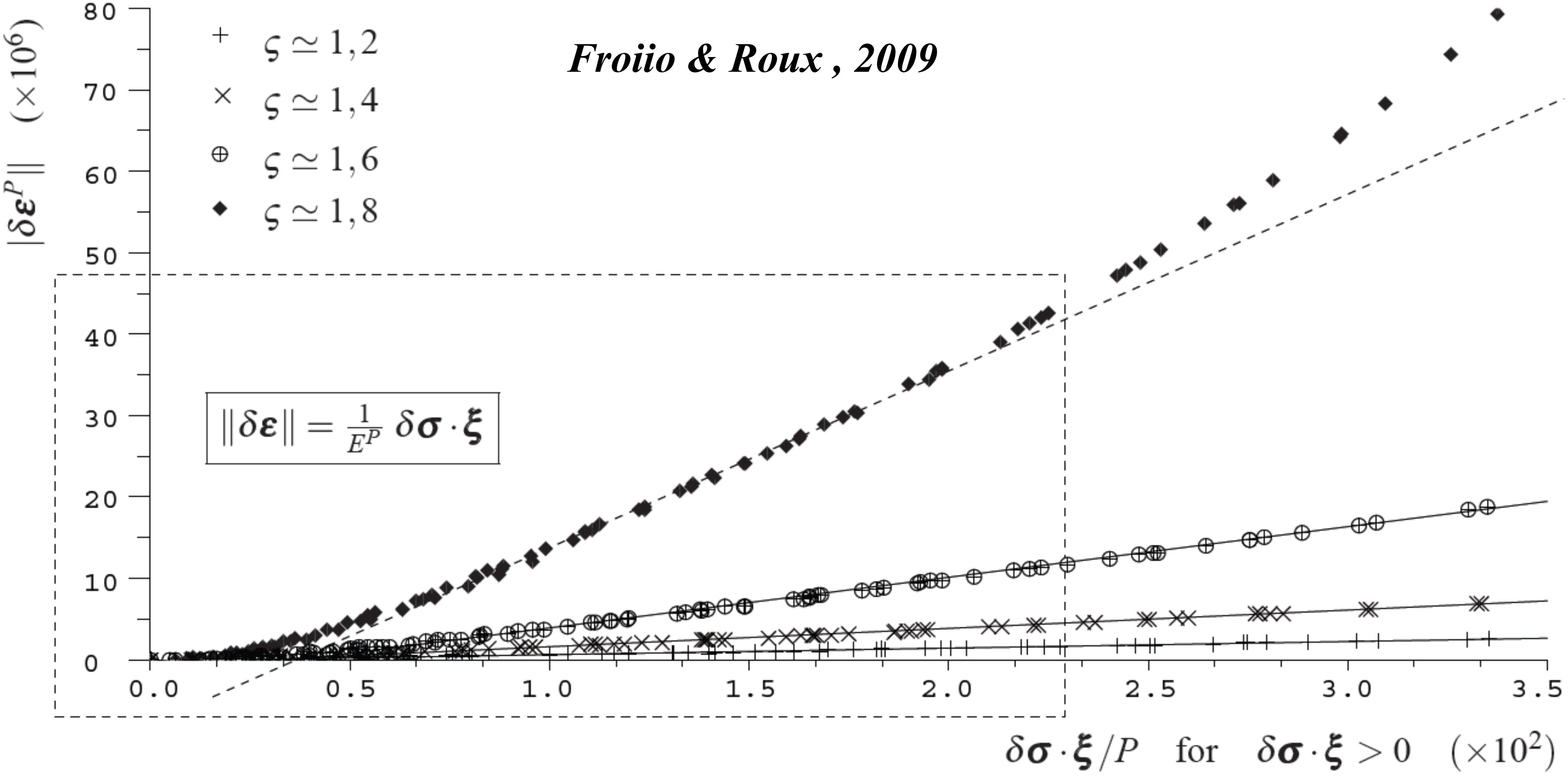


Sibille, 2006

- Irreversible mechanisms, contact sliding, opening (eventually discontinuous in time) involved for sufficiently large stress increments.

IV. How to perform stress or strain probes with DEM

IV-2 Stress loading increment



- According to elasto-plasticity, the plastic strain increment is proportional to the active part of the stress increment (part of the stress increment pointing outward from the elastic domain).

⇒ to respect this condition: $\|d\,\vec{\sigma}\| < 0.022\,P$ ... but not too small!

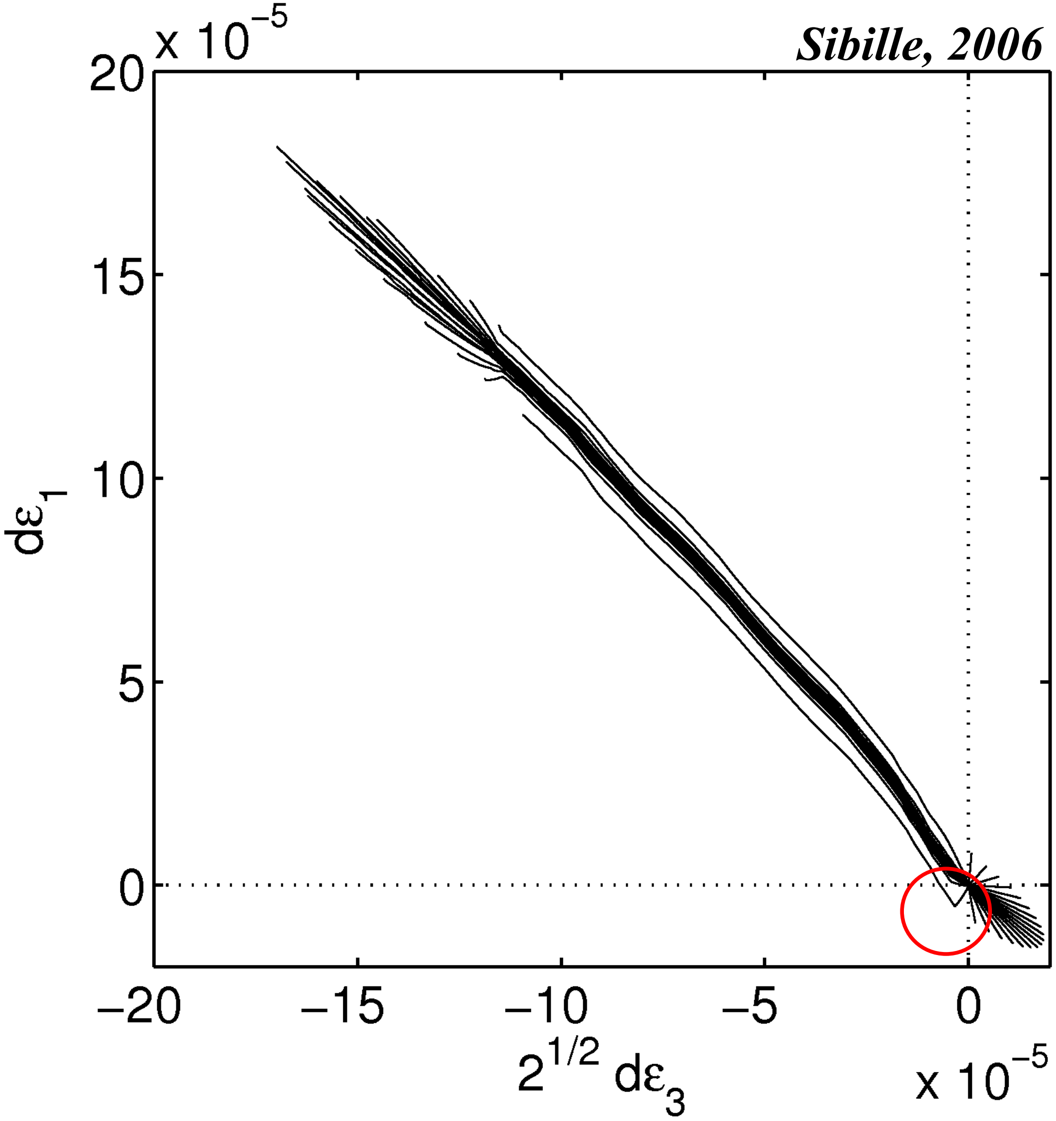
IV. How to perform stress or strain probes with DEM

IV-2 Stress loading increment

⇒ existence of residual elastic response from the initial state considered.

“ A small parasite effect of this intermediate `creep' transition [during the stabilisation of the sample at the initial stress state] before stress probing is that part of the plastic memory, stored at contact between particles, is erased due to a slight unavoidable rearrangement of the contact network ” (Froiio & Roux, 2009)

Sibille, 2006



IV. How to perform stress or strain probes with DEM

IV-2 Stress loading increment

⇒ **What size of stress loading increment should we choose?**

A size sufficiently small to characterize the initial stress state considered with its proper history (and not other stress states in the vicinity), but sufficiently large to involved irreversible mechanisms (contact sliding, opening, contact creation) characterizing the stress state considered.

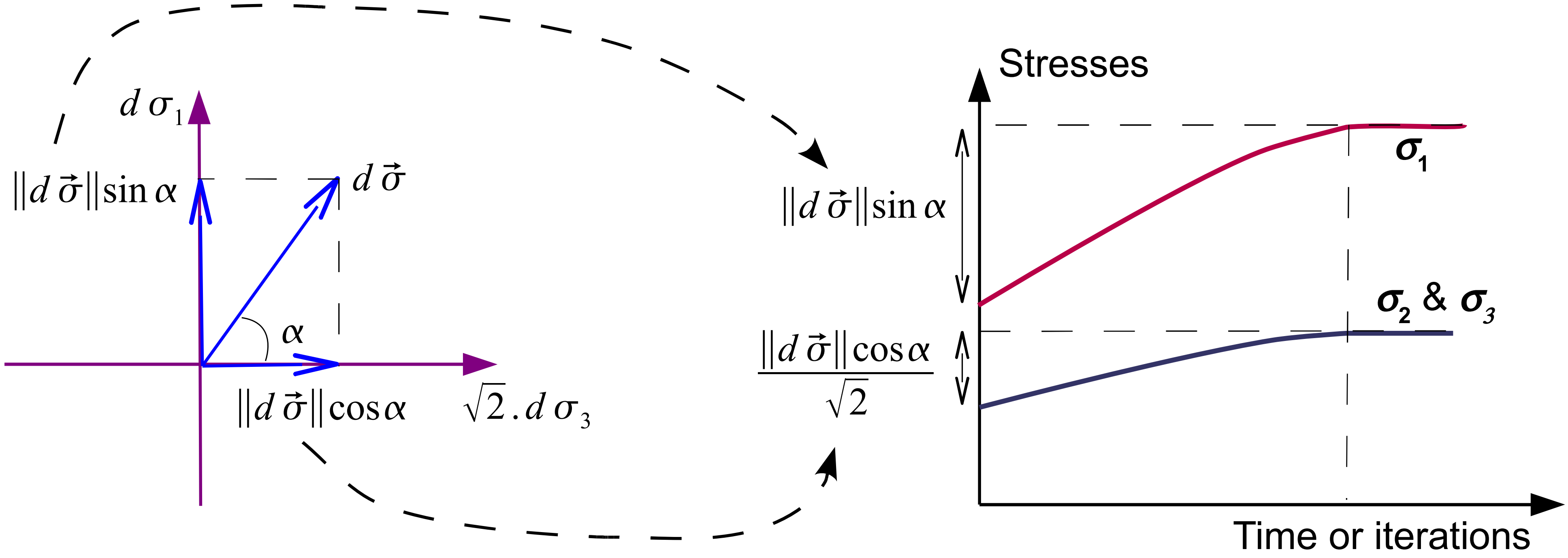
Authors	**Type**	$\|d\vec{\sigma}\|$ **(kPa)**	$\|d\vec{\sigma}\|/p_0$
Bardet	DEM 2D	-	0.05
Royis & Doanh	Exp. tests	10	0.10
Calvetti *et al.*	DEM 3D	10	0.10
Kishino	DEM 3D	1	0.01
Alonso-Marroquin	DEM 2D	0.016	10^{-4}
Sibille	DEM 3D	1	0.01
Froiio & Roux	DEM 2D	-	<0.02 (but not too small)

⇒ **when** $\kappa = \dfrac{k_n}{\langle D_S \rangle p}$ ↗ **then** $\|d\vec{\sigma}\|/p_0$ ↘ (Froiio & Roux, 2009)

IV. How to perform stress or strain probes with DEM

IV-3 Apply the stress increment in a direction α

⇒ Increase progressively each principal stresses such that a rectilinear stress path is followed (i.e. such that the final value is reached at the same time for σ_1 and σ_3)



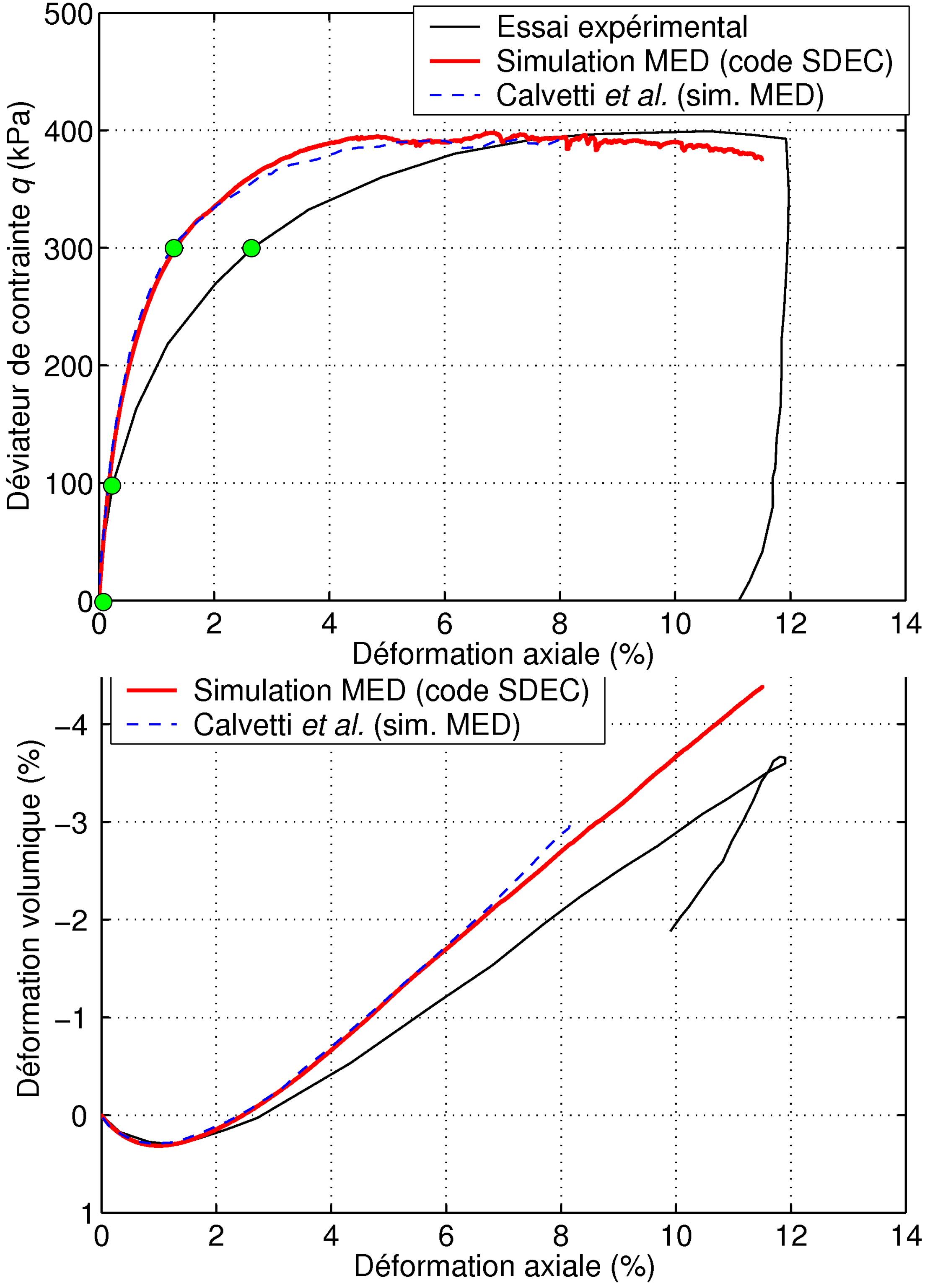
```
ThreeDTriaxialEngine(stressControl_1=1,  stressControl_2=1, stressControl_3=1,
   sigma1=Sa_curr, sigma2=Sr_curr, sigma3=Sr_curr,
   strainRate1=100, strainRate2=100, strainRate3=100)
```

With:

```
Sa_curr = Sa_final * nbIte / nbIteRamp
Sr_curr = Sr_final * nbIte / nbIteRamp
```

IV. How to perform stress

IV-3 Apply the stress increm



<u>Typical strain response envelops:</u>

- DEM simulation by *Calvetti et al.*:

Calvetti F., Viggiani G., Tamagnini C., “ A numerical investigation of the incremental behavior of granular soils”, Rivista Italiana di Geotecnica, vol. 3, p. 11-29, 2003

and Sibille (2006)

(spherical particles with locked rotations, purely frictional contacts)

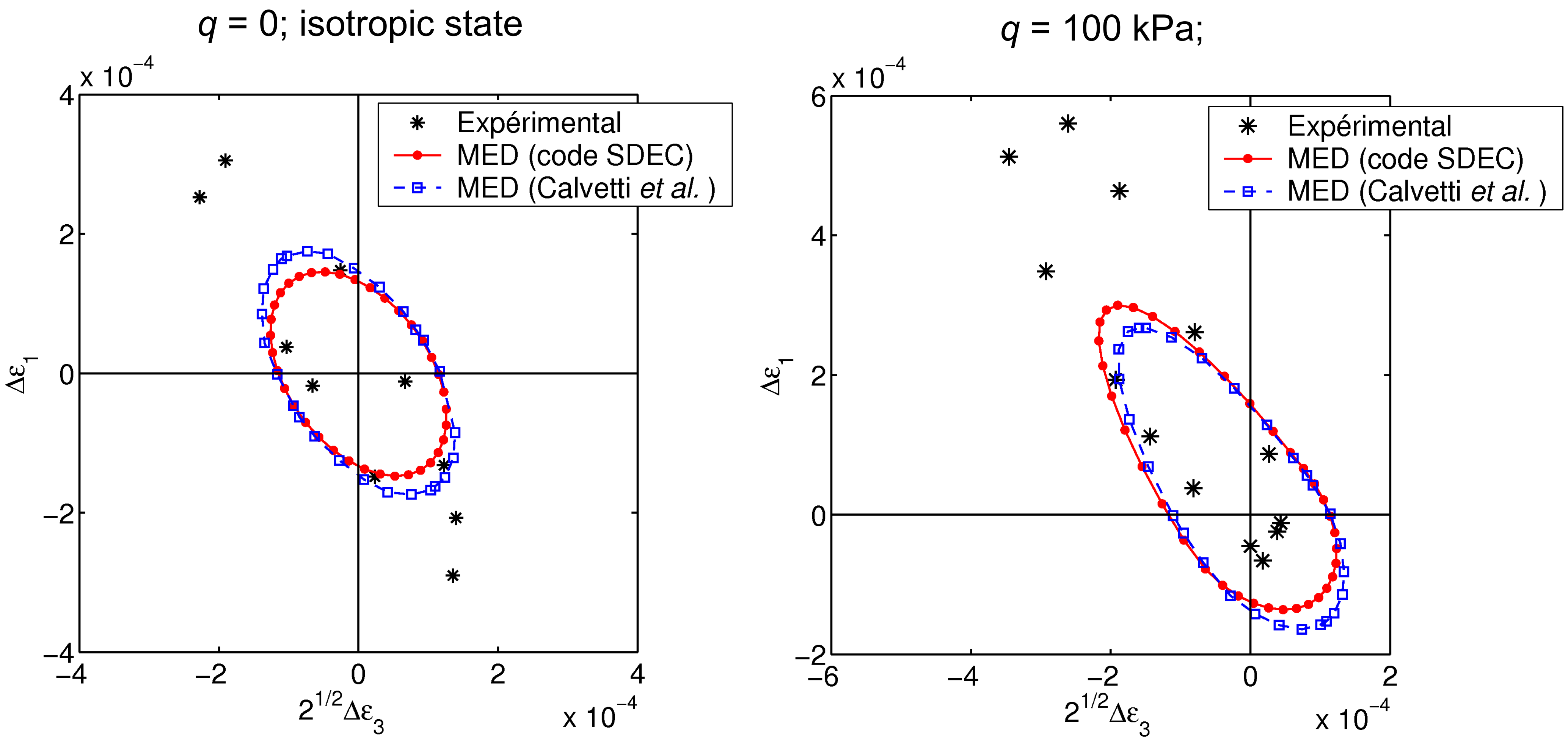
- DEM model fitted on experimental results on dense Hostun sand from *Royis & Doanh*:

Royis P. and Doanh T., “Theoretical analysis of strain response envelopes using incrementally non-linear constitutive equations”, IJNAMG, vol. 22, p. 97-132, 1998.

⇒ Stress probes at 3 stress deviator levels: q = 0; 100 and 300 kPa

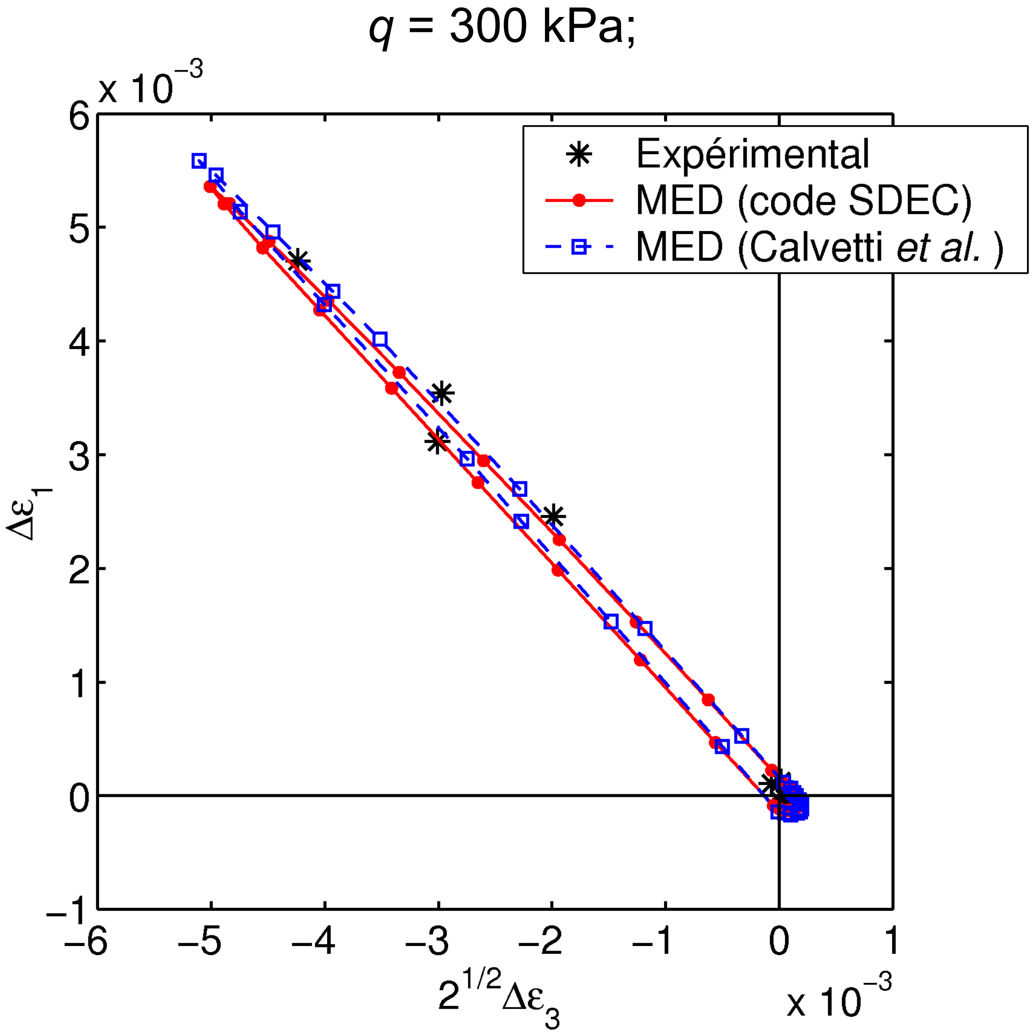
IV. How to perform stress or strain probes with DEM

IV-3 Apply the stress increment in a direction α



II. How to perform stress or strain probes with DEM

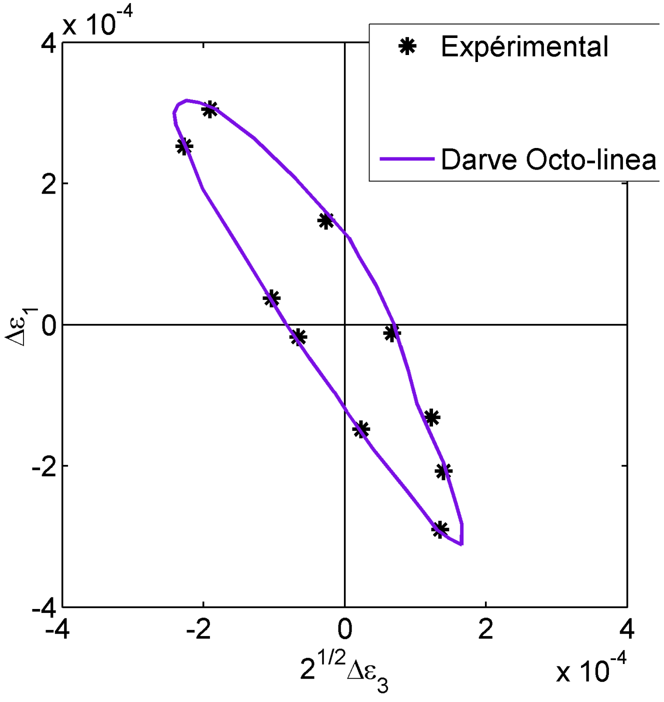
IV-3 Apply the stress increment in a direction α

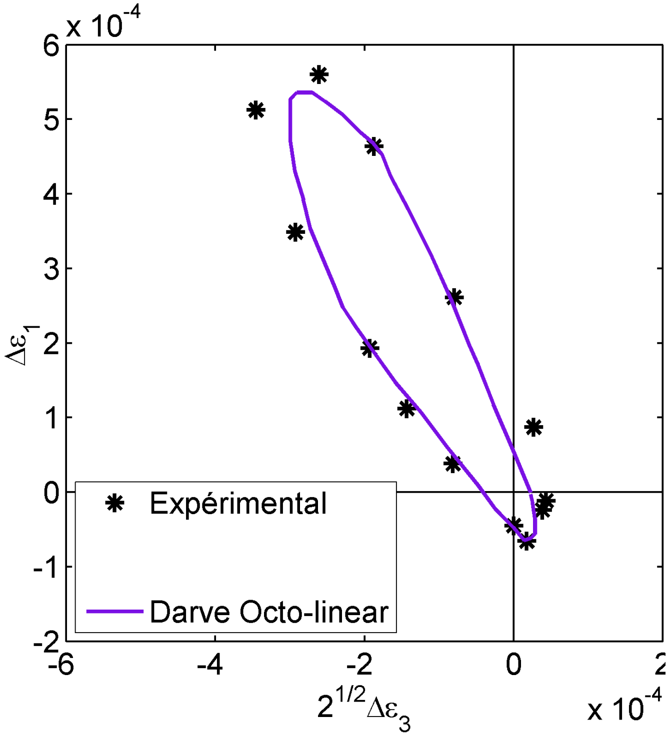


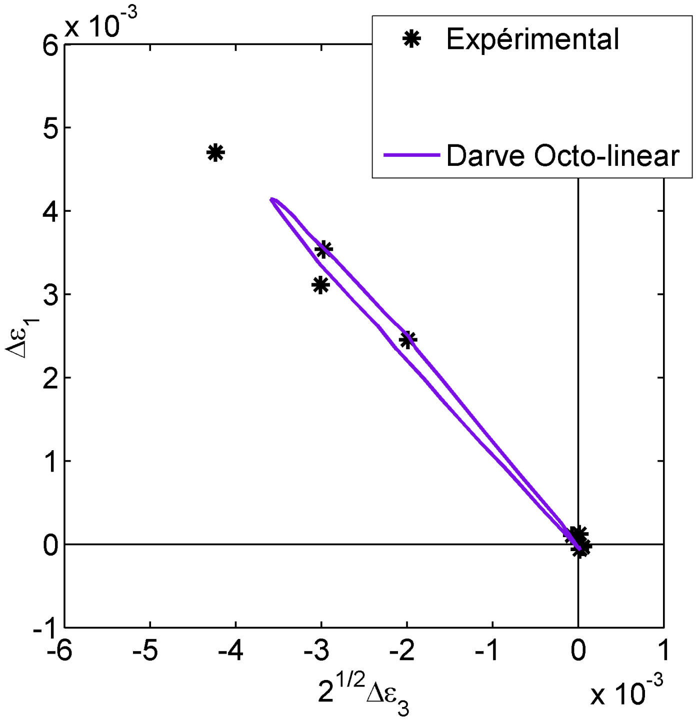
IV. How to perform stress or strain probes with DEM

IV-3 Apply the stress increment in a direction α

Same experimental results, but comparison with an incrementally piece-wise linear constitutive relation: Darve's Octolinear model (8 tensorial zones with respectively height linear relations between stress and strain increments)

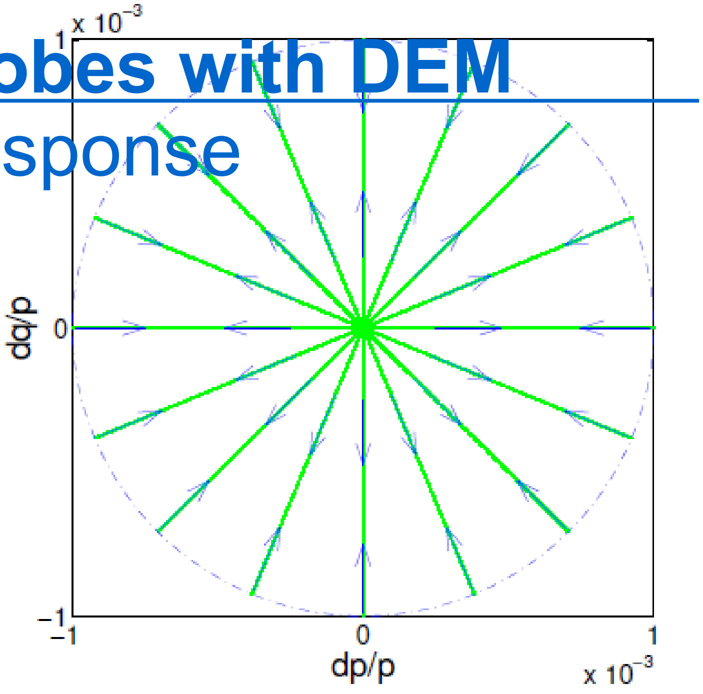






IV. How to perform stress or strain probes with DEM

IV-4 Reversible and irreversible strain response



The strain response envelop can be split into:

- **a reversible strain response envelop**
- **an irreversible strain response envelop**

We assume that: $d\vec{\varepsilon} = d\vec{\varepsilon}_r + d\vec{\varepsilon}_i$
(or in the framework of elasto-plasticity: $d\vec{\varepsilon} = d\vec{\varepsilon}_e + d\vec{\varepsilon}_p$)

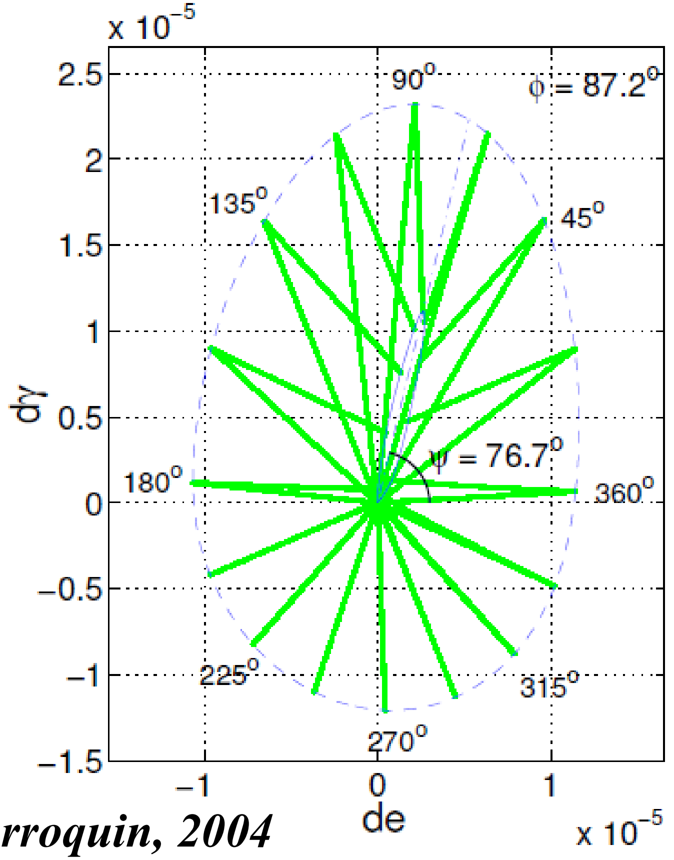
<u>Three different methods:</u>

1/ For each probe direction perform (*Bardet 1994, Kishino 2003, Alonso-Marroquin 2004*):

• an incremental loading by applying $d\vec{\sigma}$ → computation of the total strain response

• then unload the sample to reach the initial state considered → computation of the irreversible strain.

⇒ Hypothesis: completely reversible strain response during unloading (error limited for small size of stress increment $\|d\vec{\sigma}\|$).



Alonso-Marroquin, 2004

IV. How to perform stress or strain probes with DEM

IV-4 Reversible and irreversible strain response

Three different methods:

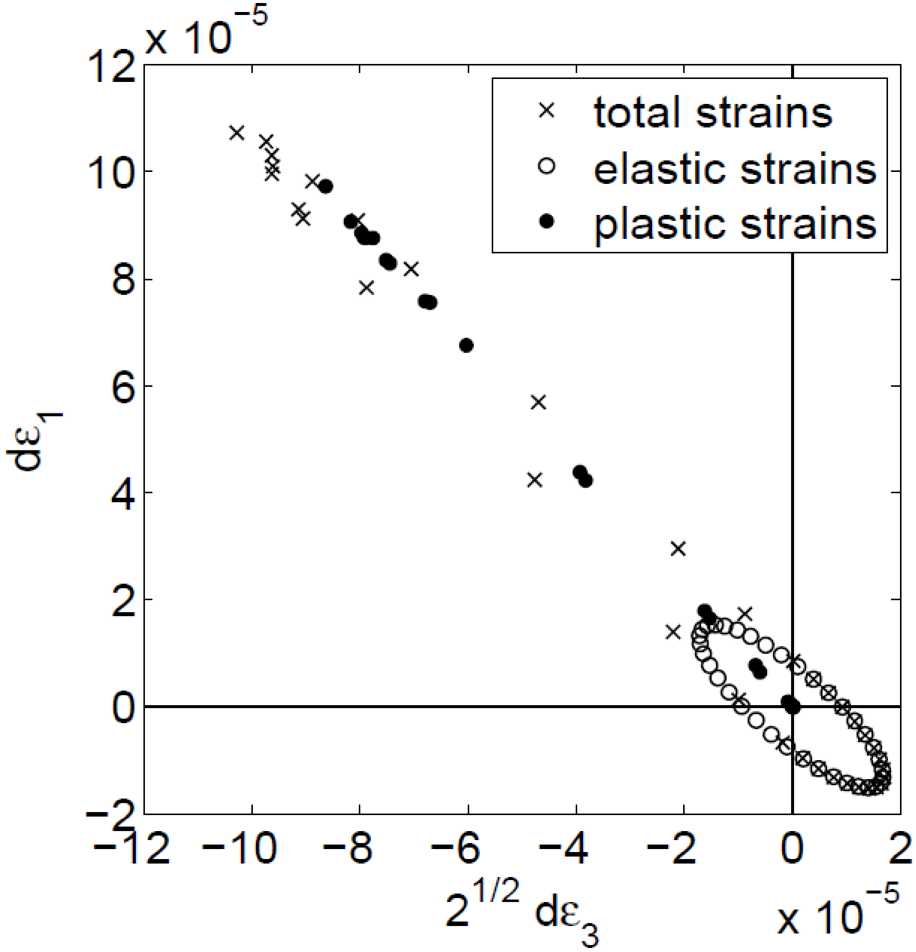
2/ For each initial stress state perform two stress probes (*Calvetti et al. 2003, Sibille et al. 2009*):

- Classical stress probes → computation of the total strain.

- Stress probes where local irreversible mechanisms are avoided (sliding, contact opening) → computation of the reversible strain.

(For inhibited particle rotations, avoiding sliding with φ_C=90° is sufficient to also prevent contact opening.)

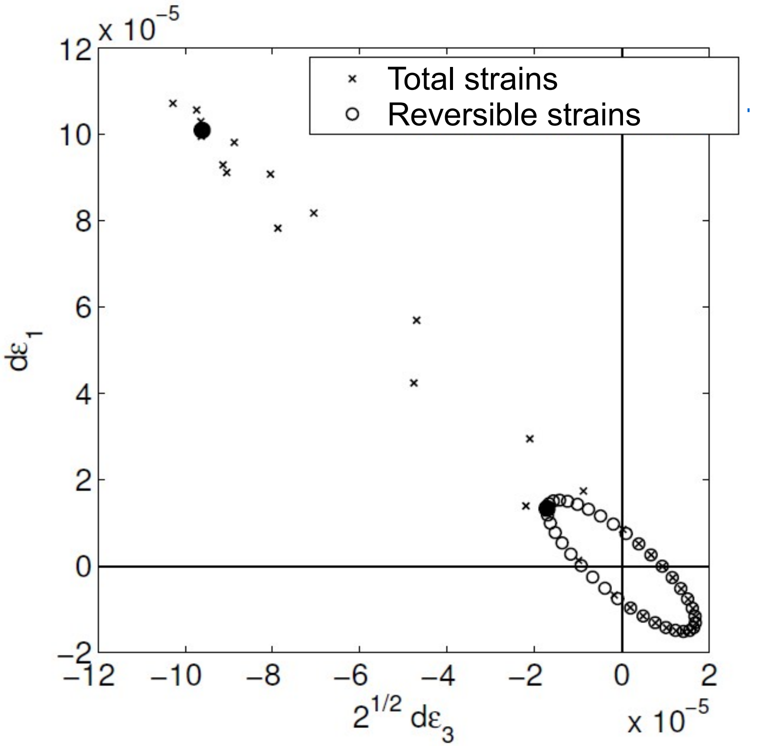
- $d\vec{\varepsilon}_i = d\vec{\varepsilon} - d\vec{\varepsilon}_r$

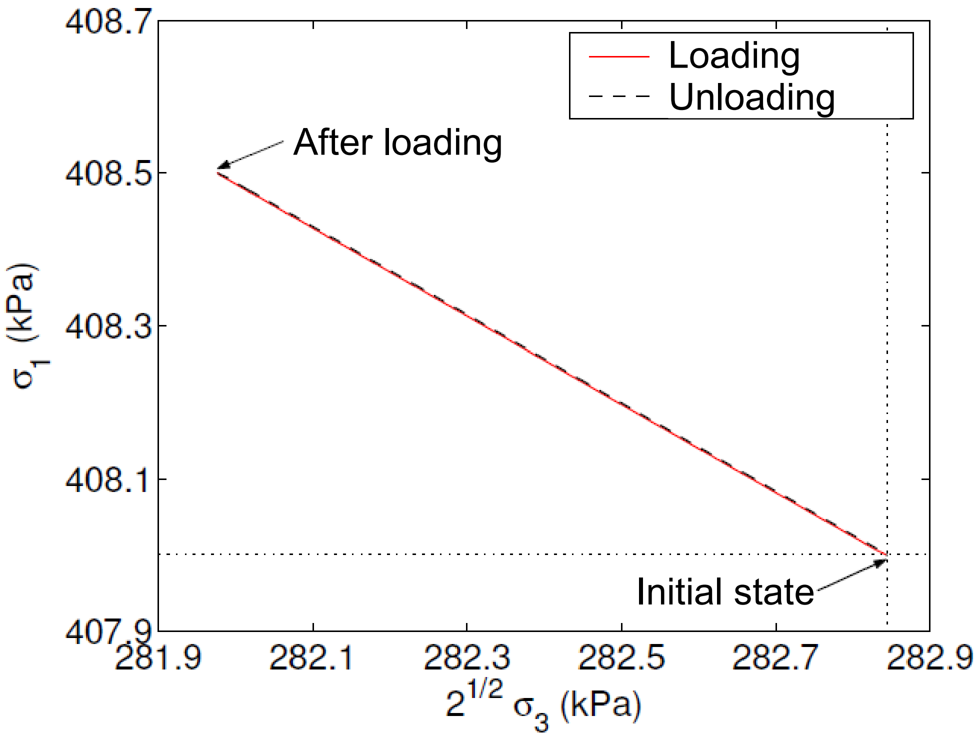


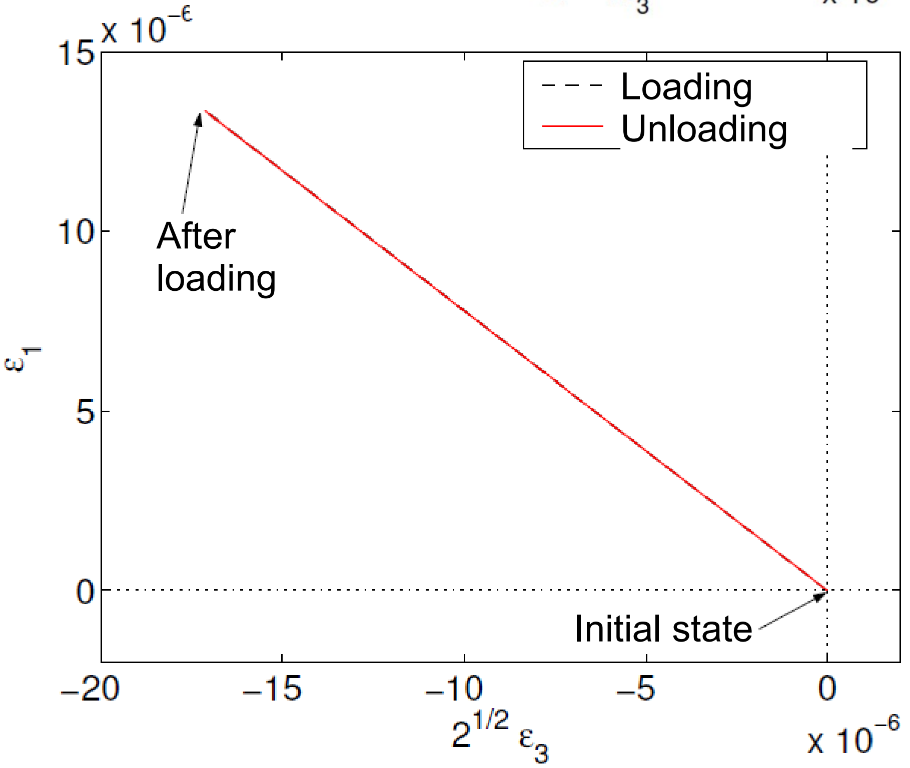
IV. How to perform stress or strain p

IV-4 Reversible and irreversible strain

⇒ Can we reasonably limit irreversible strains for rotational particles by inhibiting sliding only?







Spherical particles with purely frictional contact law with φ_C=90° ($\|d\vec{\sigma}\|/p_0=0.01$)

IV. How to perform stress or strain probes with DEM

IV-4 Reversible and irreversible strain response

Three different methods:

3/ Use the stiffness matrix associated to the contact network (*Froiio & Roux 2010*):

- Classical stress probes → computation of the total strain.

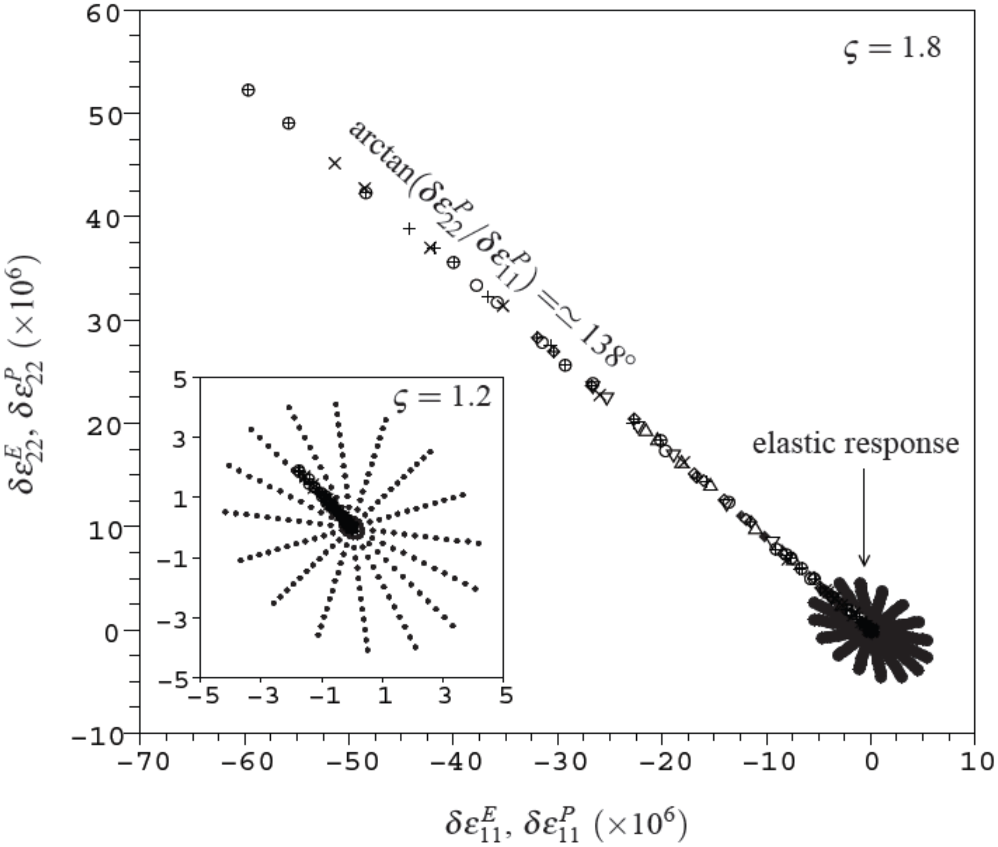
- Build the elasticity tensor $\tilde{C}_e$ ($d\vec{\sigma}=\tilde{C}_e d\vec{\varepsilon}_e$) by assembling the contributions stiffness k_n and k_t of each contact involved in the contact network (*Agnolin & Roux 2007*).

Compute the elastic part of strains from:

$$d\vec{\varepsilon}_e=\tilde{C}_e^{-1}d\vec{\sigma}$$

- $d\vec{\varepsilon}_p=d\vec{\varepsilon}-d\vec{\varepsilon}_e$

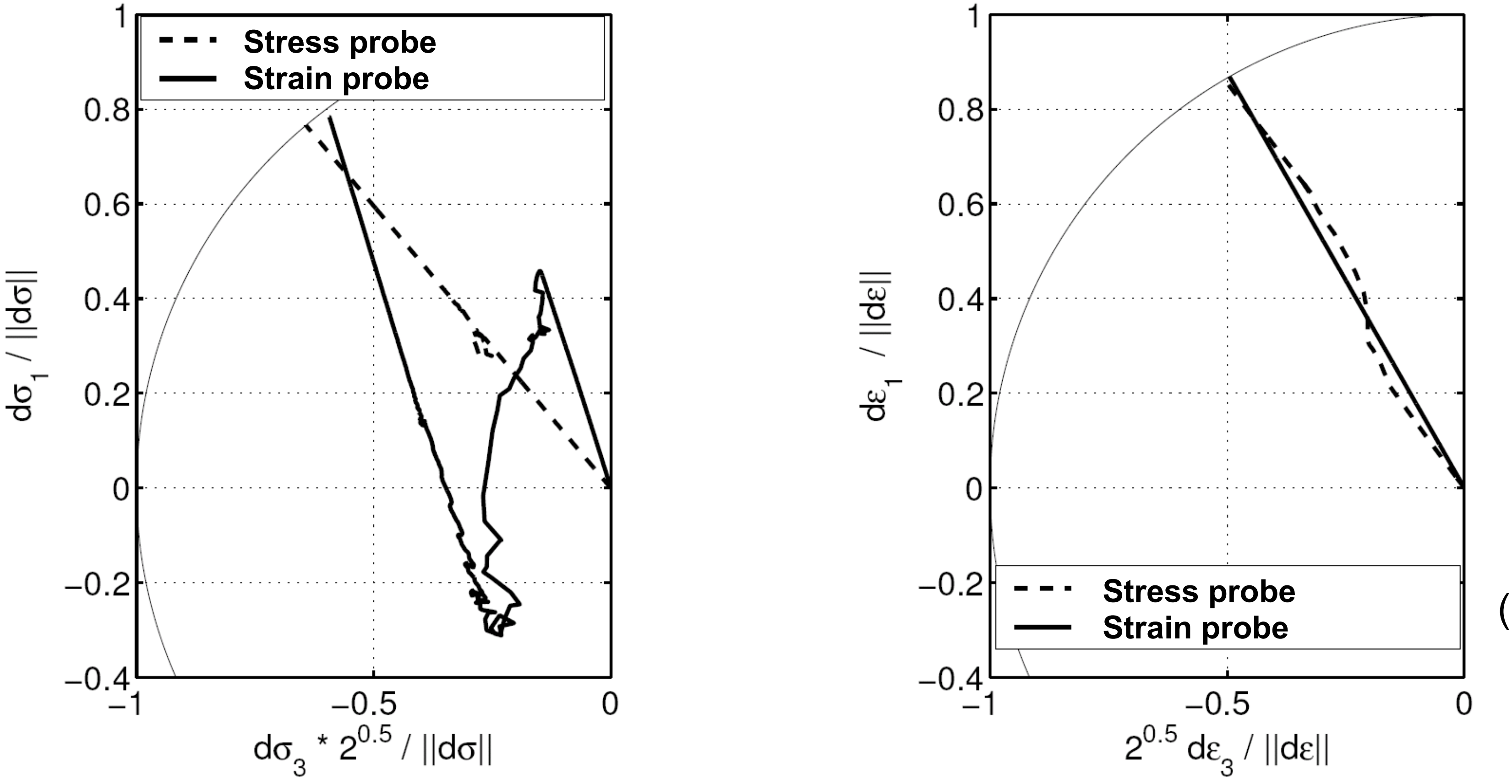
Froiio & Roux 2010



IV. How to perform stress or strain probes with DEM

IV-5 Stress or strain probes?

Stress probes and strain probes are dual → make your choice



(Sibille 2006)

Nevertheless:

- It's easier to perform strain probes with DEM (no need of stress control, fixed strain rate, stabilisation easier after the application of $d\varepsilon$)

- Interpretation in framework of elastoplasticity easier from stress probes (elastic and plastic strain decomposition...)

V. Interpretation in the framework of elastoplasticity

V-1 Few words about elastoplasticity

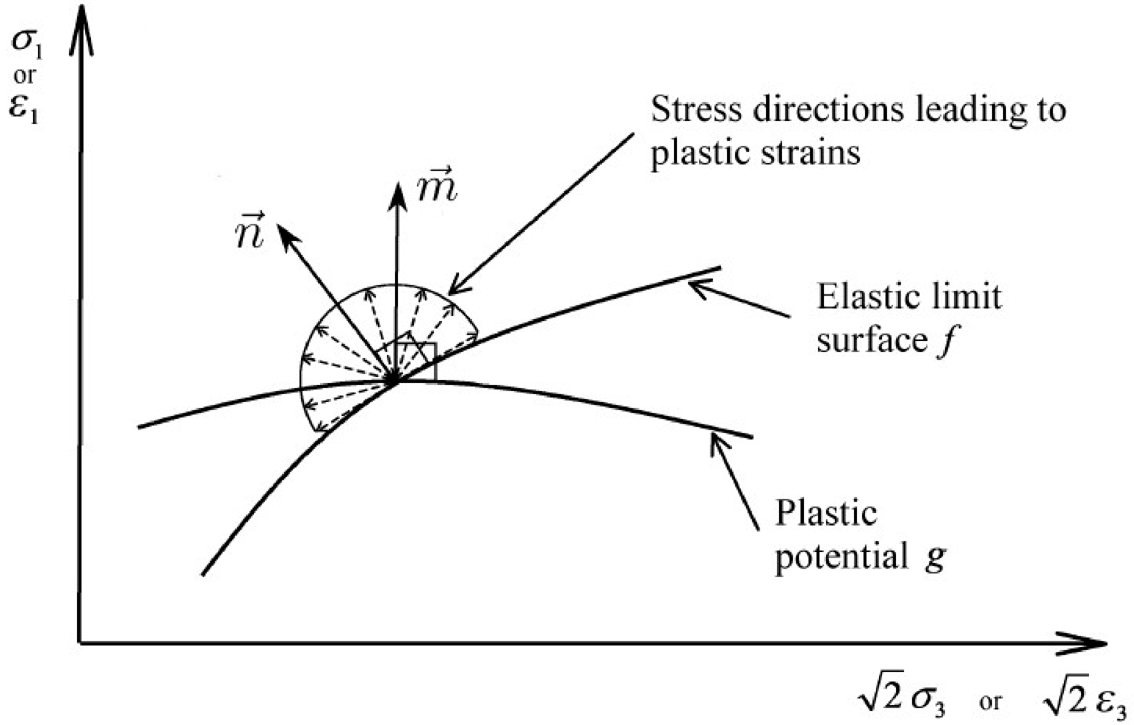
Strain decomposition into elastic strain (reversible) and plastic strain (irreversible):

$$d\tilde{\varepsilon} = d\tilde{\varepsilon}_e + d\tilde{\varepsilon}_p$$

Yield surface f (elastic limit surface): surface in the stress space limiting the stress states reach from fully reversible strains.

Plastic potential g: surface in the stress space; the increment plastic strain vector is perpendicular to the plastic potential (Flow rule).

For $g \neq f$ the material is non-associated.



V. Interpretation in the framework of elastoplasticity

V-1 Few words about elastoplasticity

2 tensorial zones (elastoplasticity with single mechanism of plastic strain)

For $f(\tilde{\sigma})<0$

or

$f(\tilde{\sigma})=0$ and $\frac{\partial f}{\partial \tilde{\sigma}} d\tilde{\sigma}<0$ (unloading)

then: $d\tilde{\varepsilon}_e = \tilde{H}(\tilde{\sigma})\, d\tilde{\sigma}$

Generalised Hooke's Law

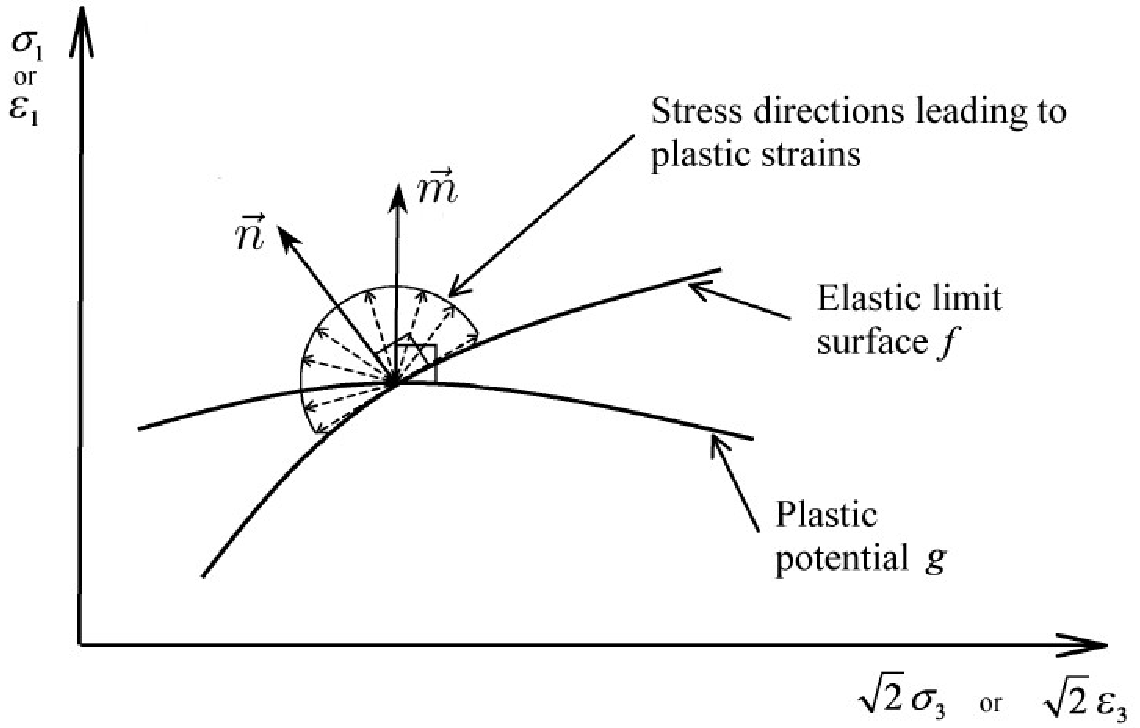
$d\tilde{\varepsilon}_p = 0$

For $f(\tilde{\sigma})=0$ and $\frac{\partial f}{\partial \tilde{\sigma}} d\tilde{\sigma}>0$ (loading)

then: $d\tilde{\varepsilon}_e = \tilde{H}(\tilde{\sigma})\, d\tilde{\sigma}$

$d\tilde{\varepsilon}_p = d\lambda \frac{\partial g}{\partial \tilde{\sigma}}$

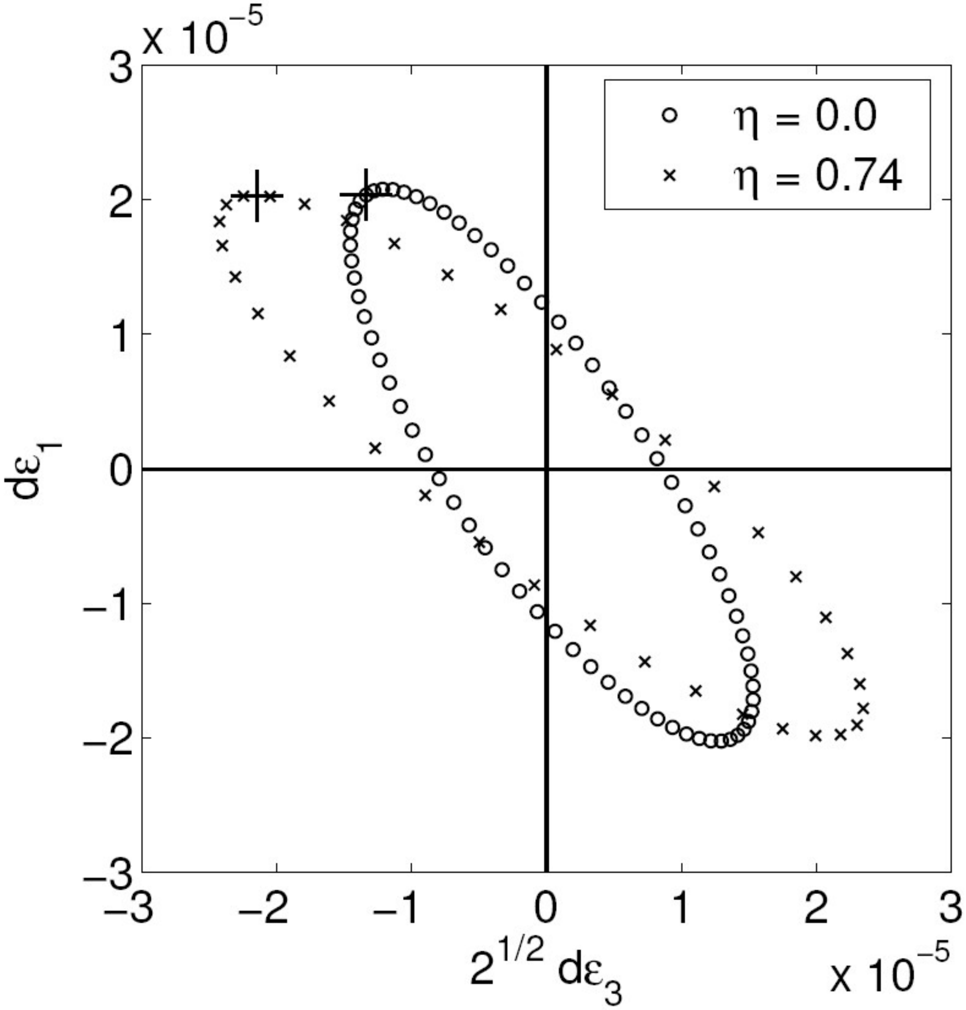
Plastic Multiplier

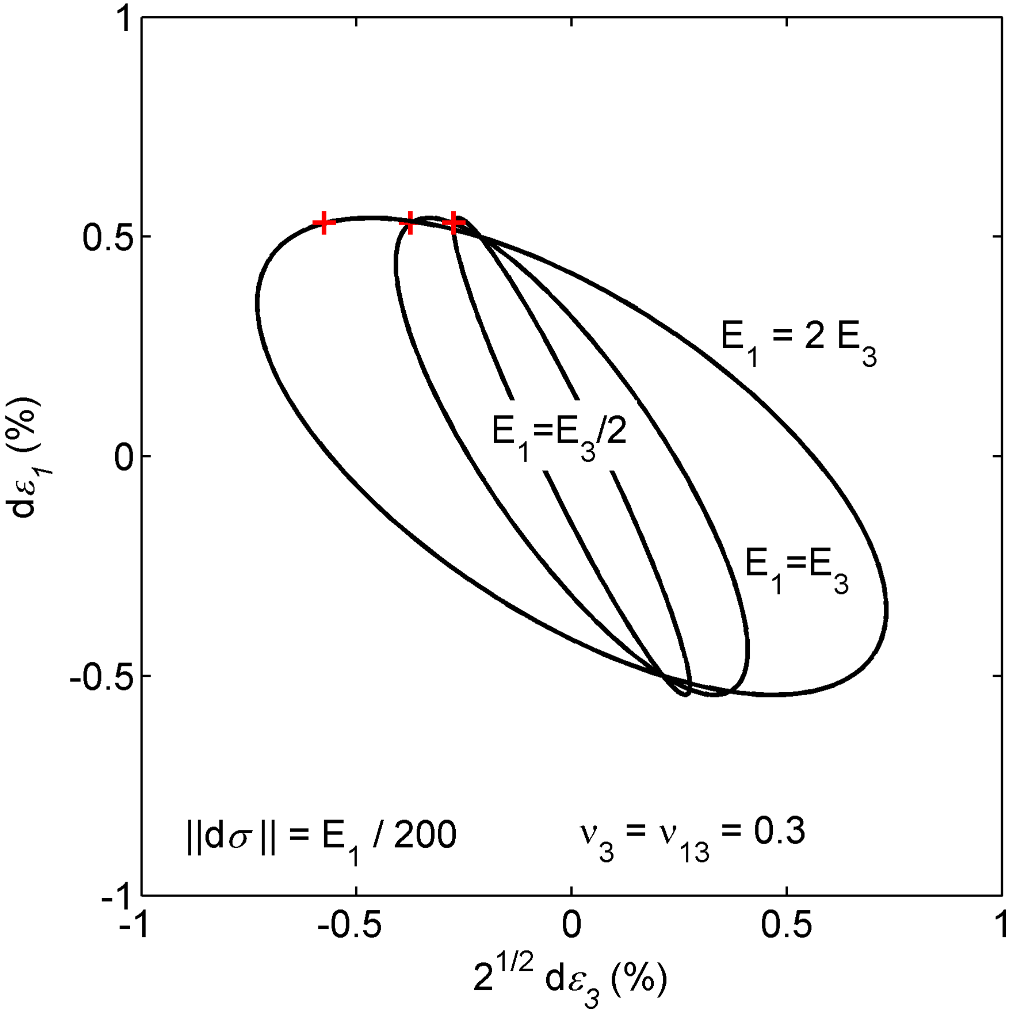


V. Interpretation in the framework of elastoplasticity

V-2 Elastic deformation

Elastic strain response envelops at the isotropic state $\eta = q / p = 0$
and for an deviatoric stress state $\eta = q / p = 0.74$





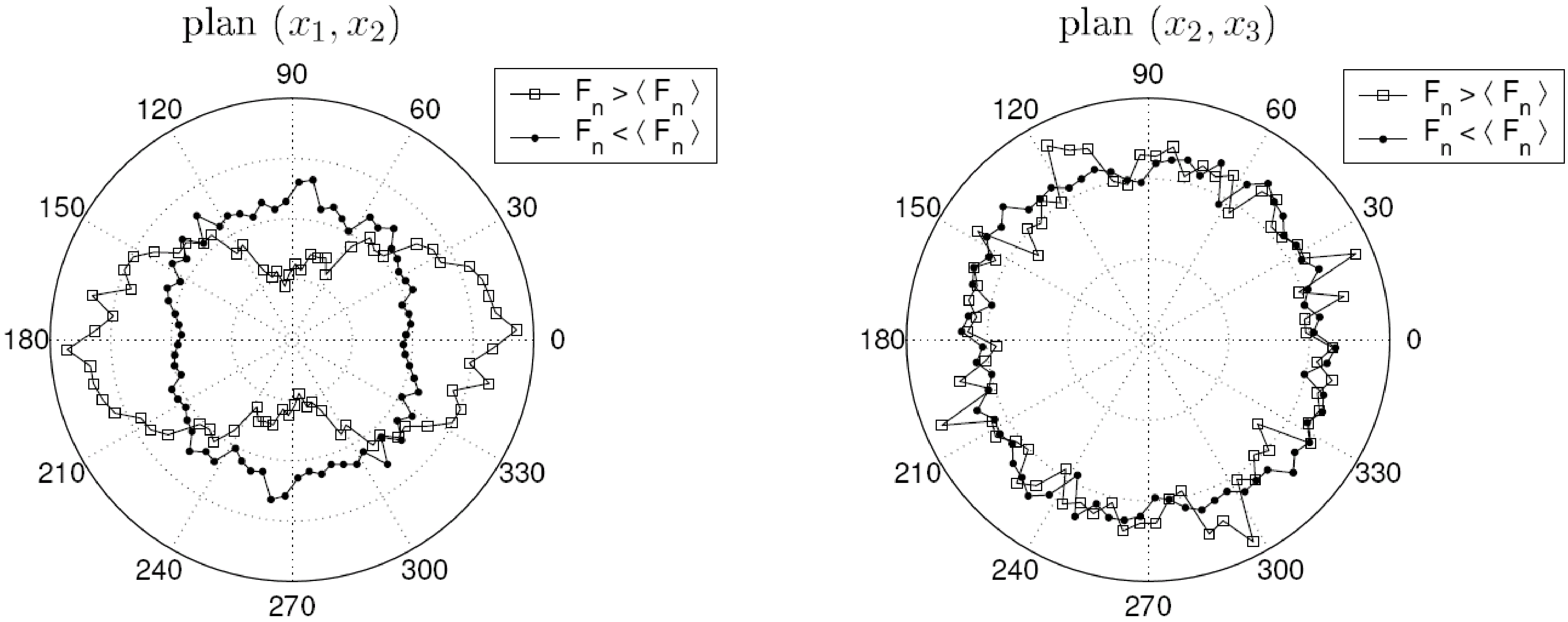
⇒ Response envelops typical of an isotropic, and transverse isotropic, elastic linear behaviour (can be modelled with a generalized Hooke's law)

V. Interpretation in the framework of elastoplasticity

V-2 Elastic deformation

Distribution of contact orientations after a drained triaxial compression

(x_1 is the direction of compression)



⇒ transverse isotropy of contact orientations ≡ transverse isotropic elasticity.

V. Interpretation in the fram

V-3 Plastic deformation

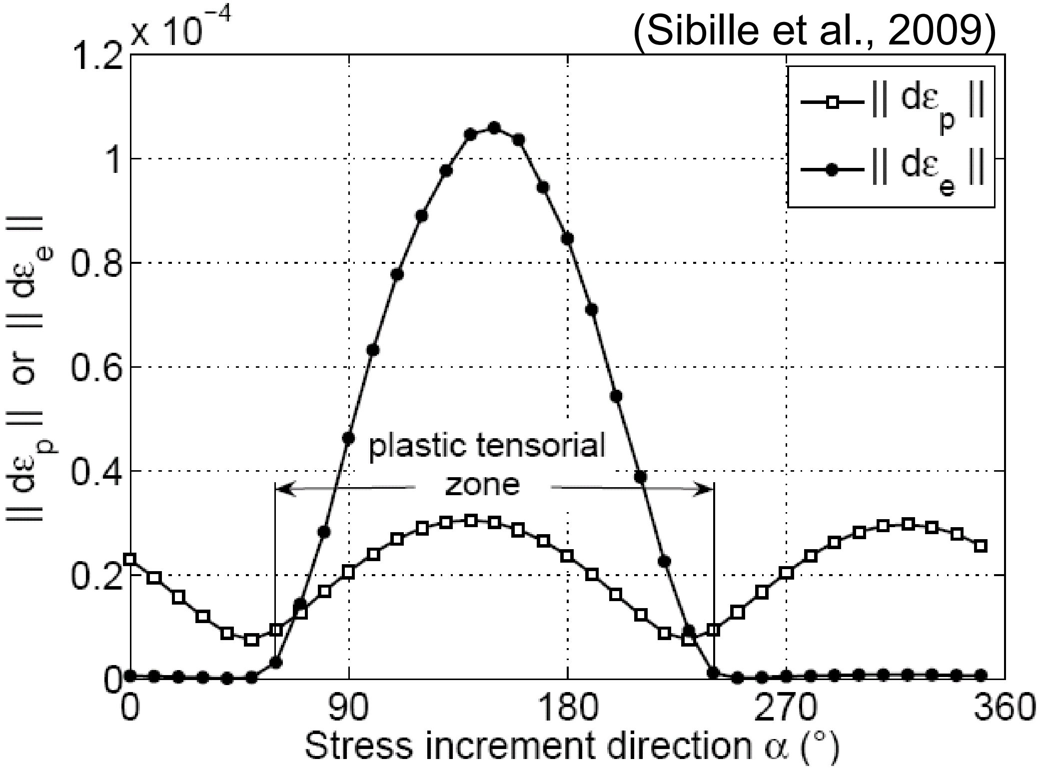
The strain response can be split into (Calvetti et al., 2003):

- the norm of the strain response vector:

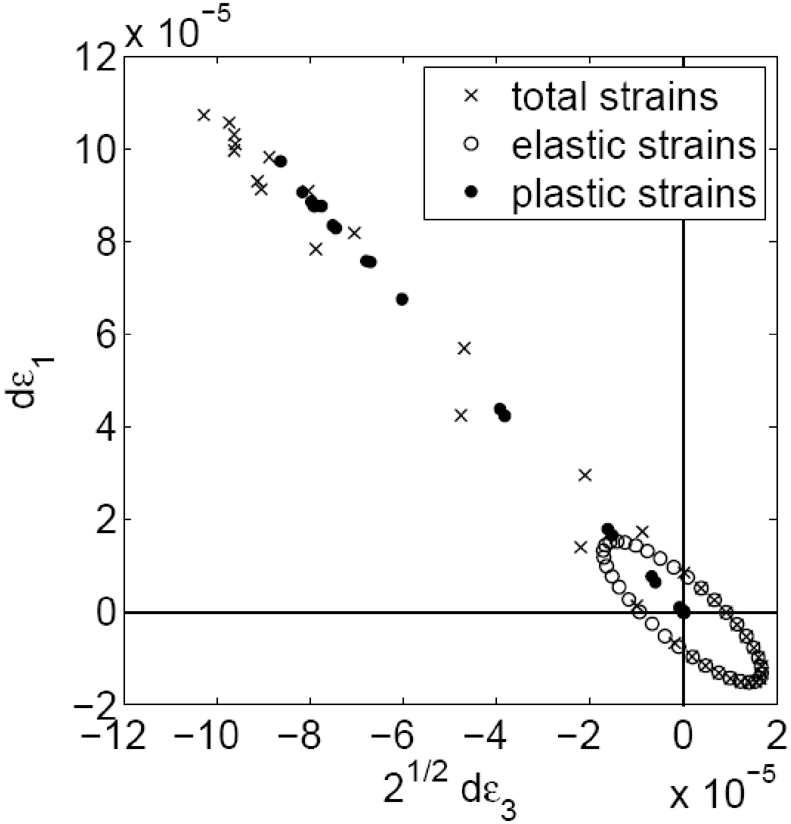
$$\|d\vec{\varepsilon}_e\|=\sqrt{d\varepsilon_{e1}^{2}+2d\varepsilon_{e3}^{2}} \qquad \|d\vec{\varepsilon}_p\|=\sqrt{d\varepsilon_{p1}^{2}+2d\varepsilon_{p3}^{2}}$$

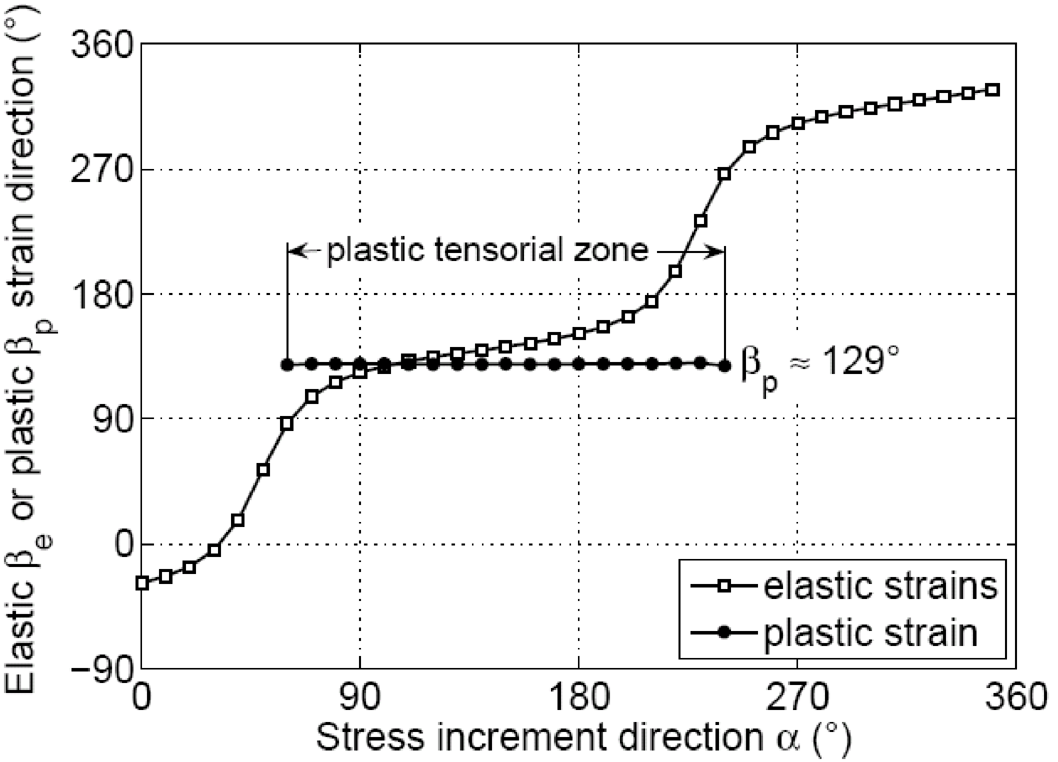
- the direction of the strain response vector:

$$\beta_e \qquad \beta_p$$



(Sibille et al., 2009)



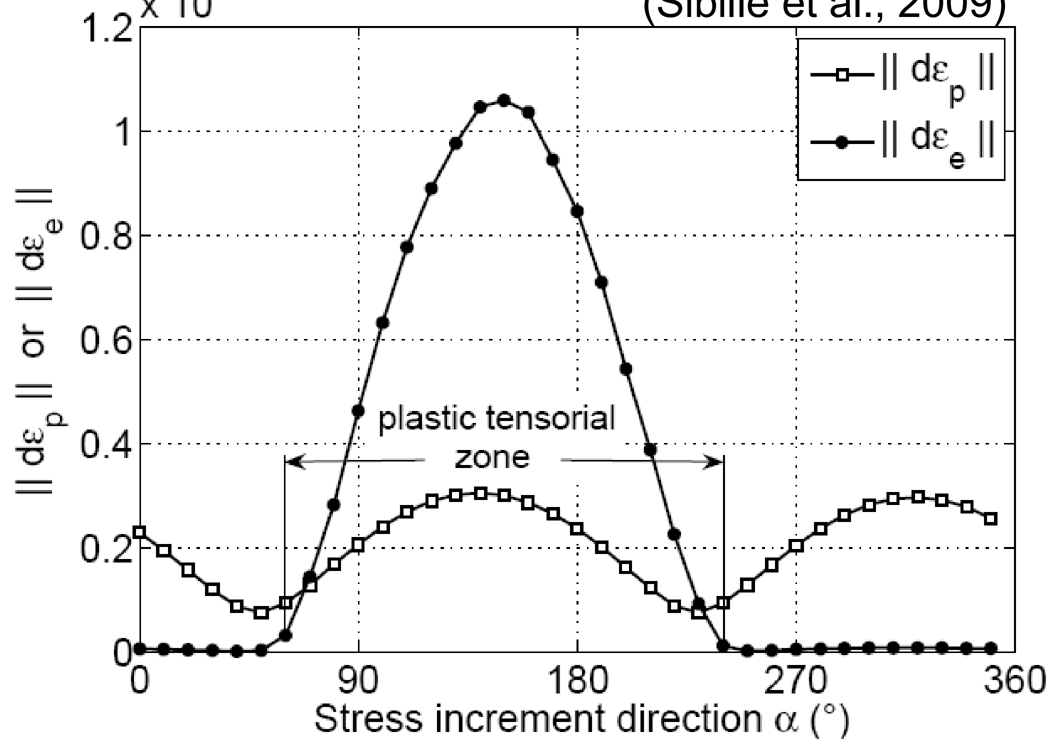


V. Interpretation in the fram

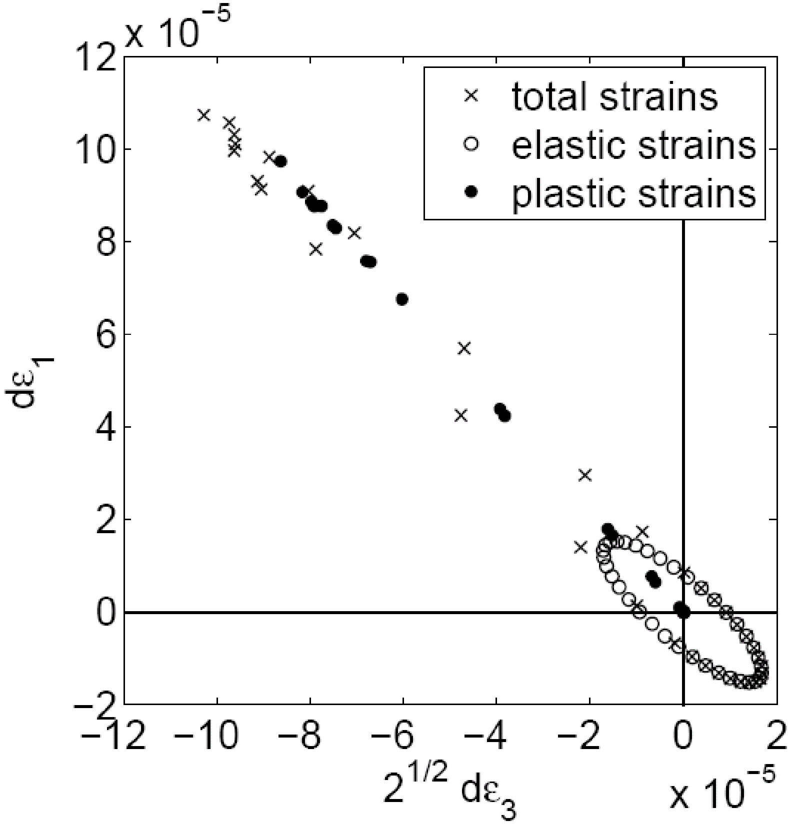
V-3 Plastic deformation

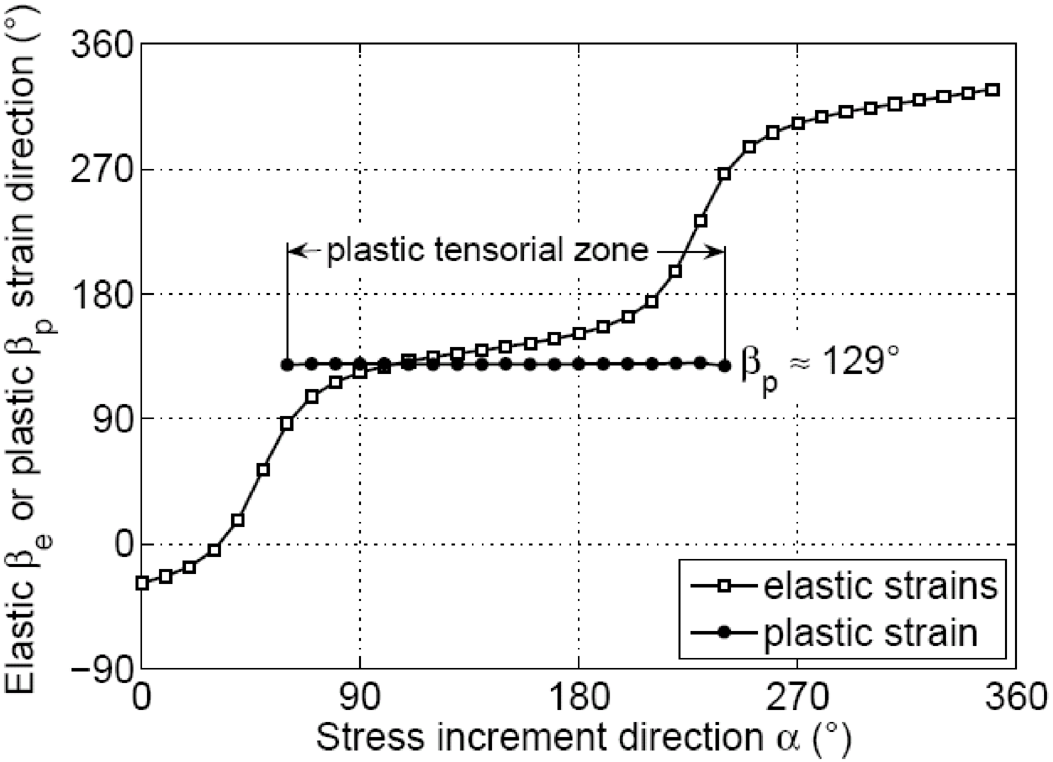
- Occurrence of plastic strains only for given stress direction: in a plastic tensorial zone.

- Direction of plastic strains constant and independent of stress direction → clear indication of flow rule existence.



(Sibille et al., 2009)





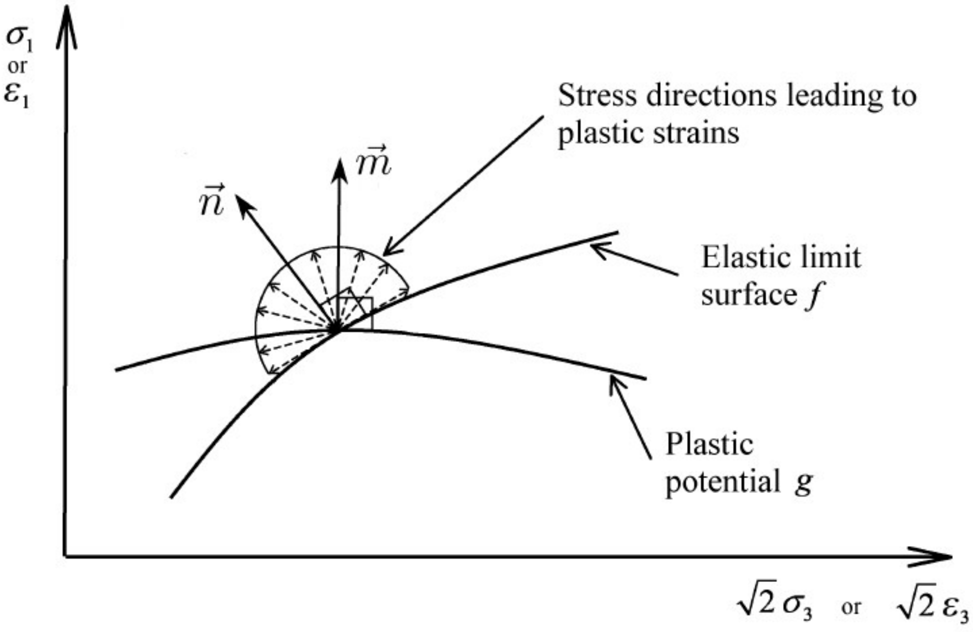
V. Interpretation in the fram

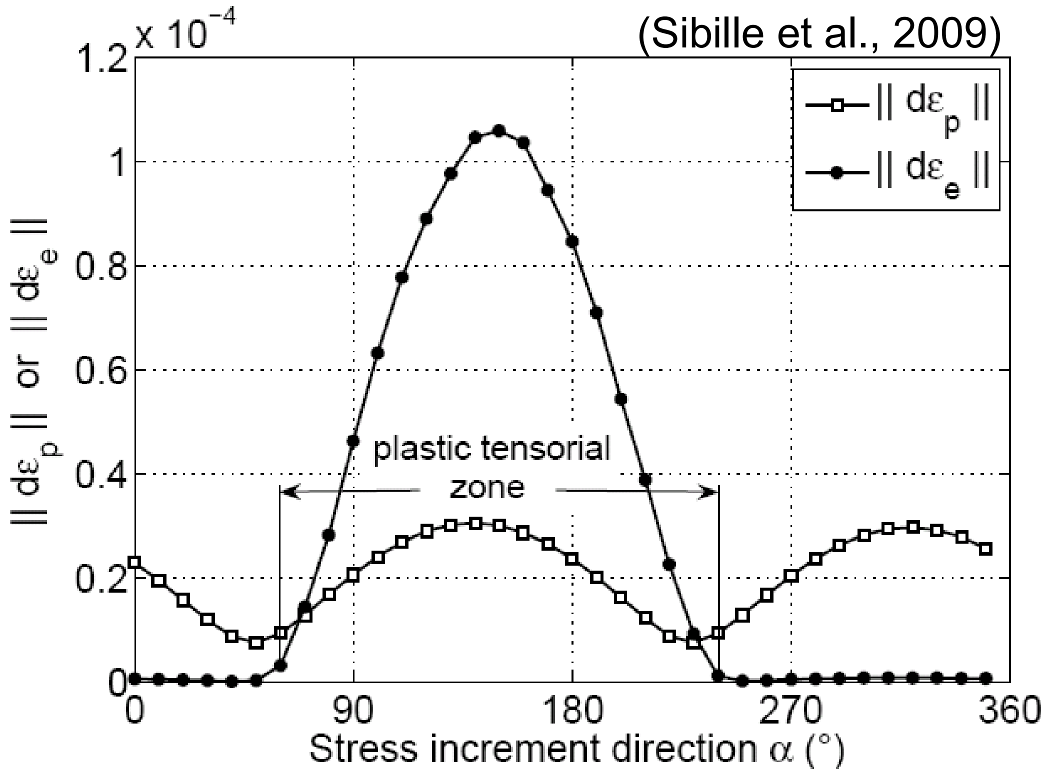
V-3 Plastic deformation

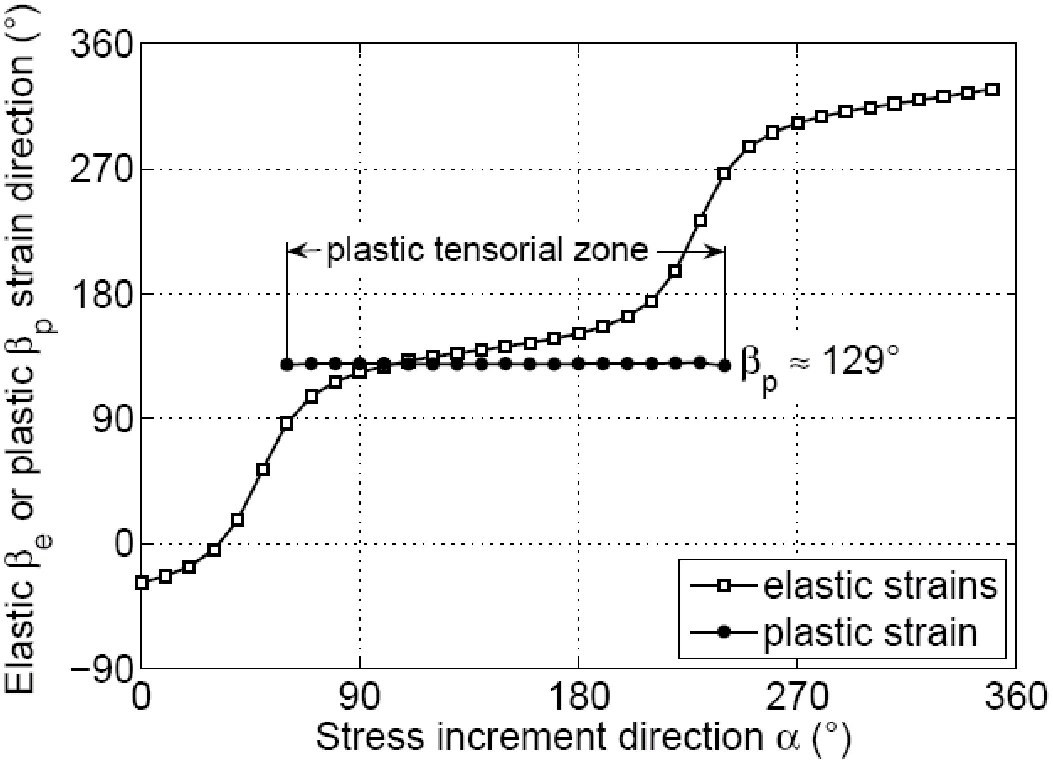
- First and last directions of plastic tensorial zone are almost tangent to the yield surface f (α = 60° and 240°).

- Rigth and left tangents are collinear (60°+180° = 240°) → smooth yield surface.

- Normal $\vec{n}$ to the yield surface along direction α = 150° → maximum of $\|d\,\vec{\varepsilon}_p\|$ ≡ $\|d\,\vec{\varepsilon}_p\|$ proportional to the active part of $d\,\vec{\sigma}$



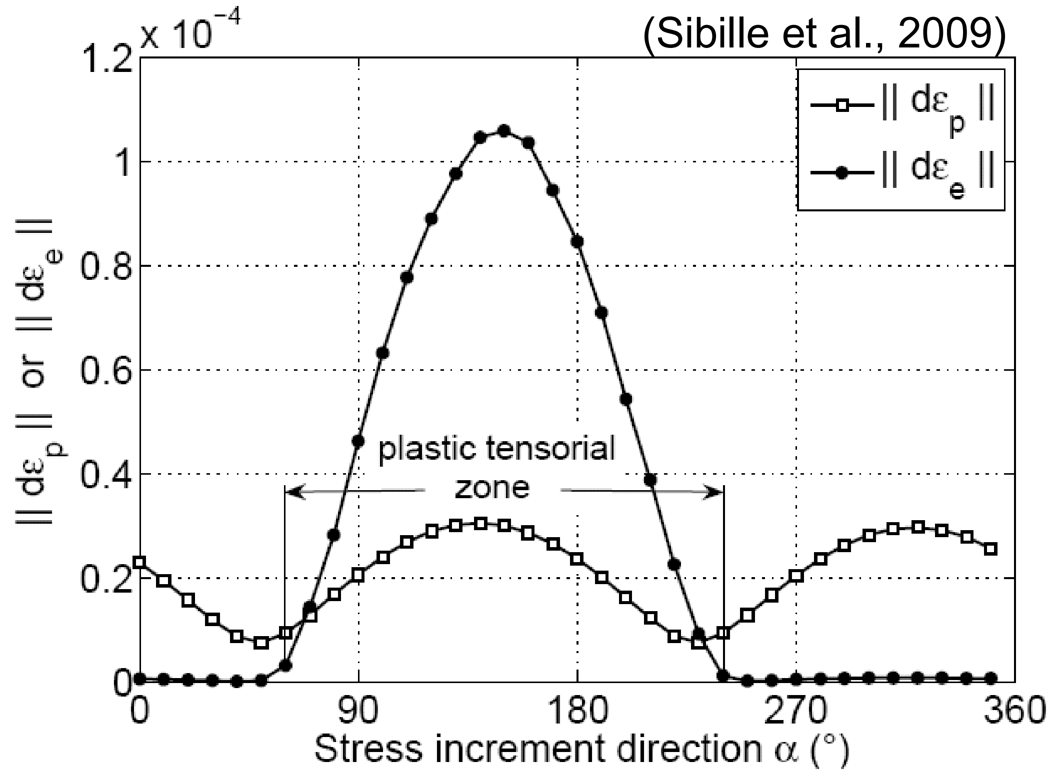

(Sibille et al., 2009)

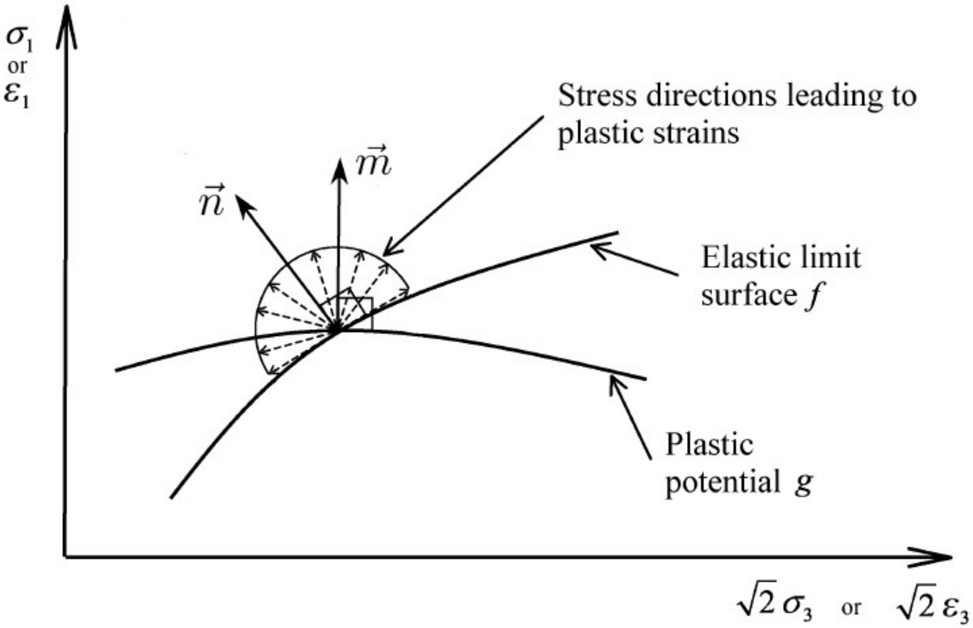


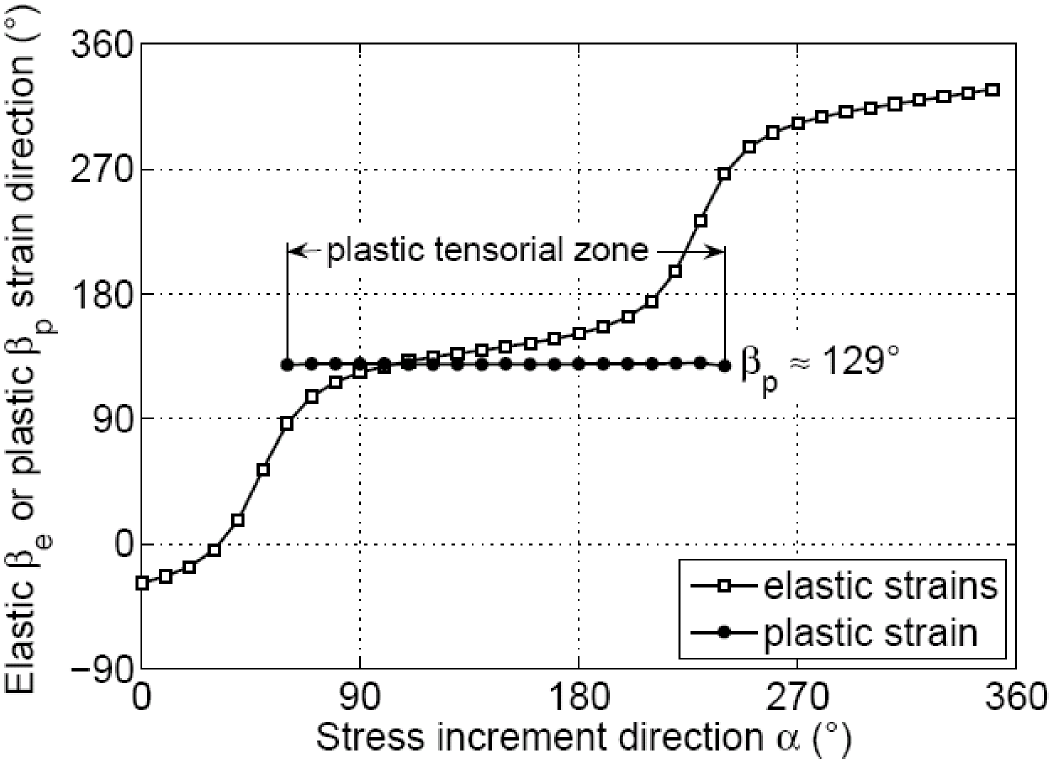
V. Interpretation in the fram

V-3 Plastic deformation

- Plastic flow in direction β_p = 129° = direction of the normal $\vec{m}$ to the flow rule
- $\vec{n}$ (150°) ≠ $\vec{m}$ (129°) → non-associated flow rule.


(Sibille et al., 2009)





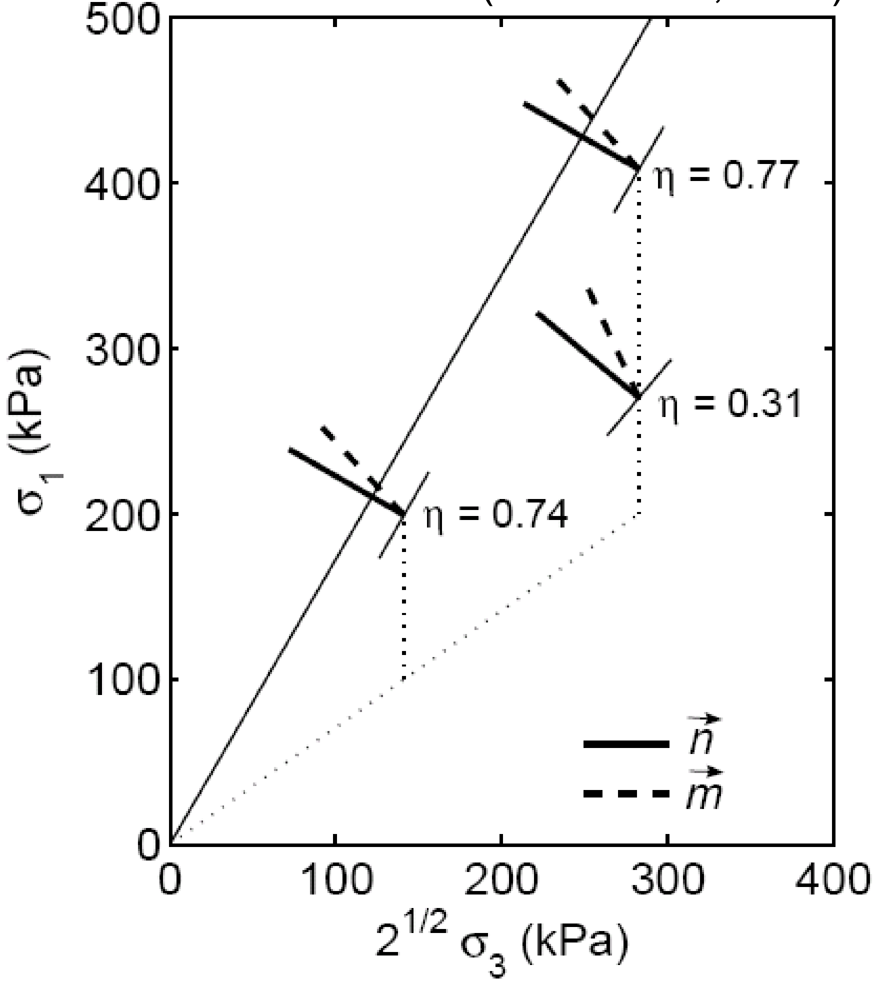
V. Interpretation in the framework of elastoplasticity

V-3 Plastic deformation

For axisymmetric stress probes after an axisymmetric initial triaxial compression:

- Elastic linear behaviour typical of a Hooke's law.
- Occurrence of plastic strains for a given tensorial zone limited by a smooth yield surface.
- Plastic flow characterized by a constant direction different from the normal to the yield surface.
- Size of plastic strain response proportional to the active part of the stress increment.

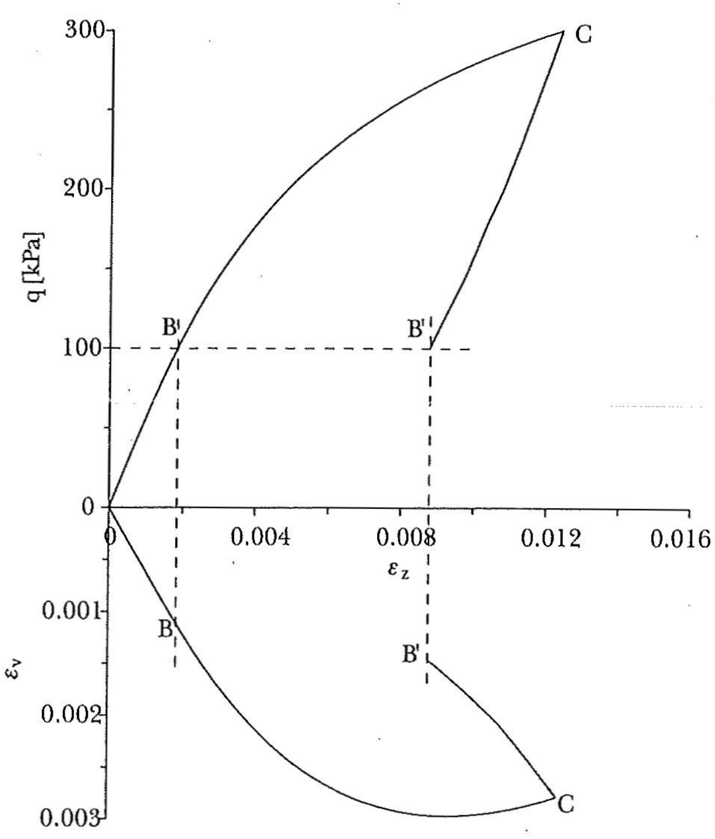
(Sibille et al., 2009)

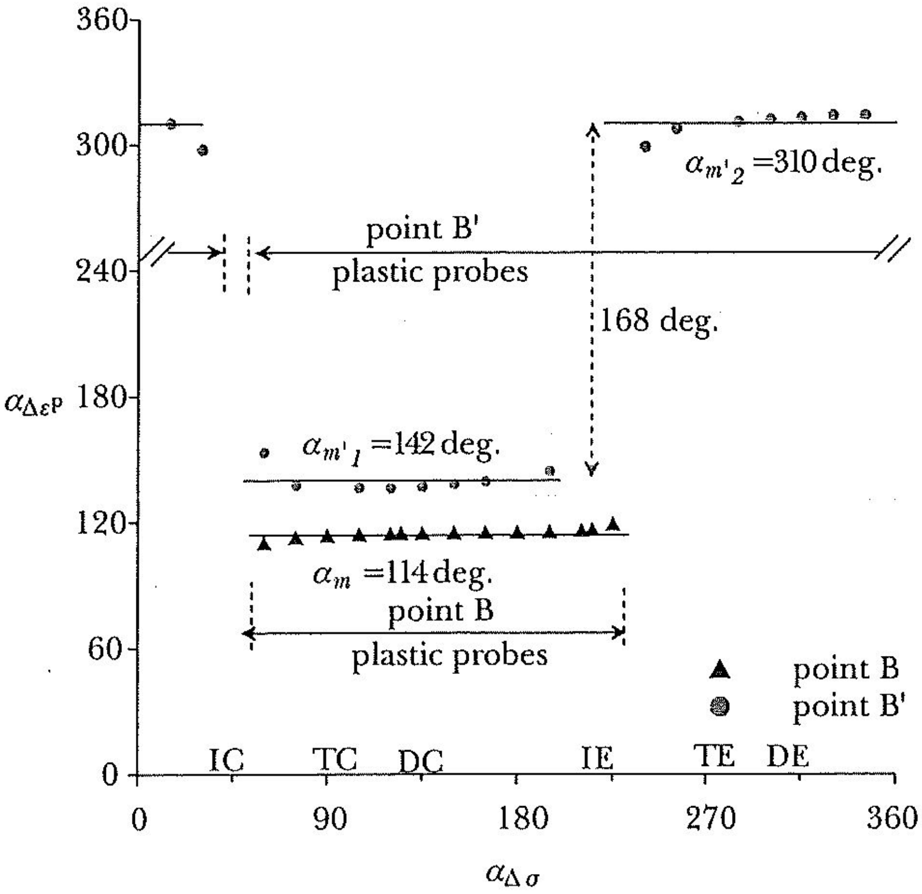


⇒ The behaviour in axisymmetric conditions of the discrete numerical assembly of spheres is very well represented by classical elasto-plasticity with a single loading mechanism.

V. Interpretation in the framework of elastoplasticity

V-4 Preloaded initial state (Calvetti et al., 2003)



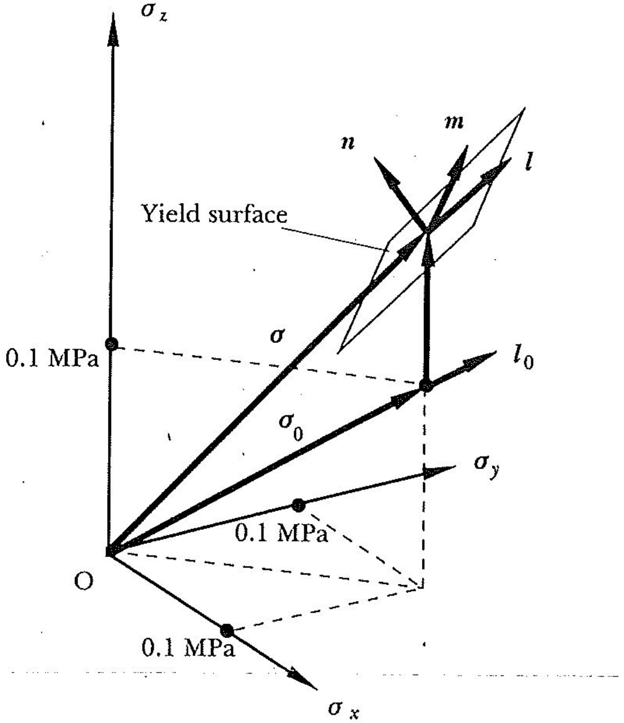


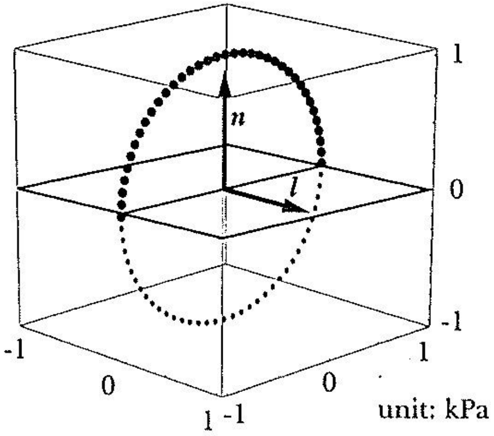
- 2 plastic flow directions (142° and 310°), in almost opposite directions!
- No purely elastic tensorial zone, but two elasto-plastic tensorial zones

⇒ cannot be described by classical elasto-plasticity!

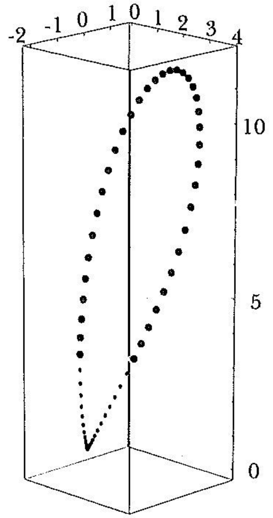
V. Interpretation in the framework of elastoplasticity

V-5 Non-axisymmetric Stress probing (Kishino, 2003)

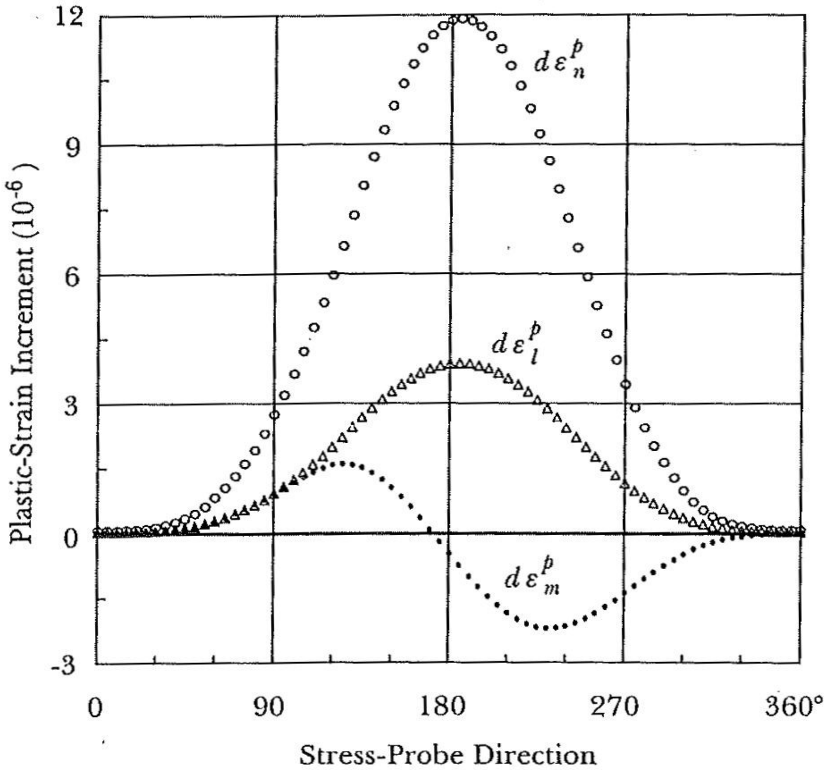




Stress probe



Plastic response envelop



- The plastic strain response envelop is not a straight line and seems to depend on stress direction.
- The “elastic” tensorial zone is well reduced (≈ 60°) → the yield surface is not flat (or not unique)
- $d\varepsilon^p_m$ synchronized with $d\sigma.m$ (and not $d\sigma.n$)

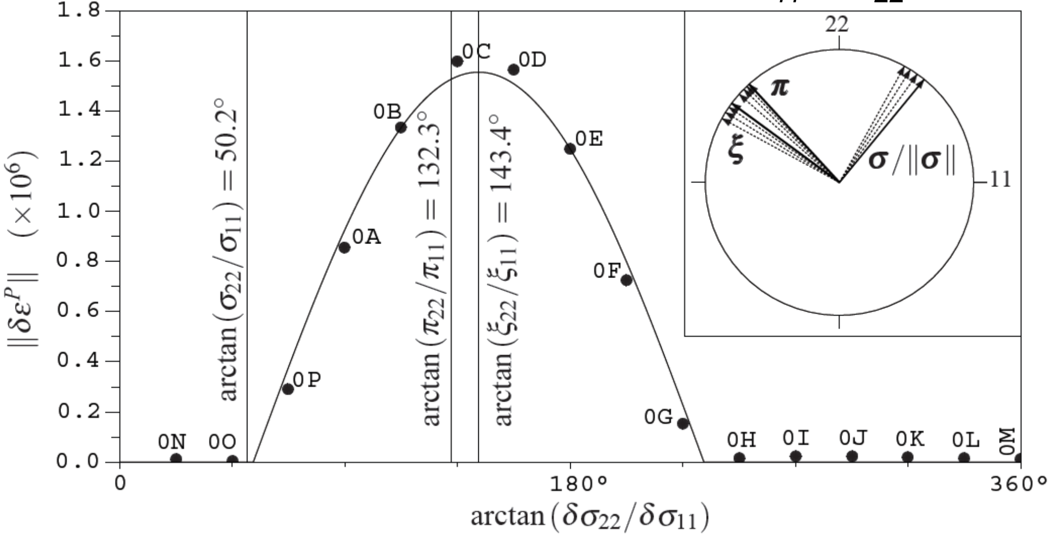
⇒ Mechanical behaviour greatly incrementally non-linear (many tensorial zones are required to describe such behaviour)

V. Interpretation in the framework of elastoplasticity

V-5 Rotation of principal stress axes (Froiio & Roux 2009)

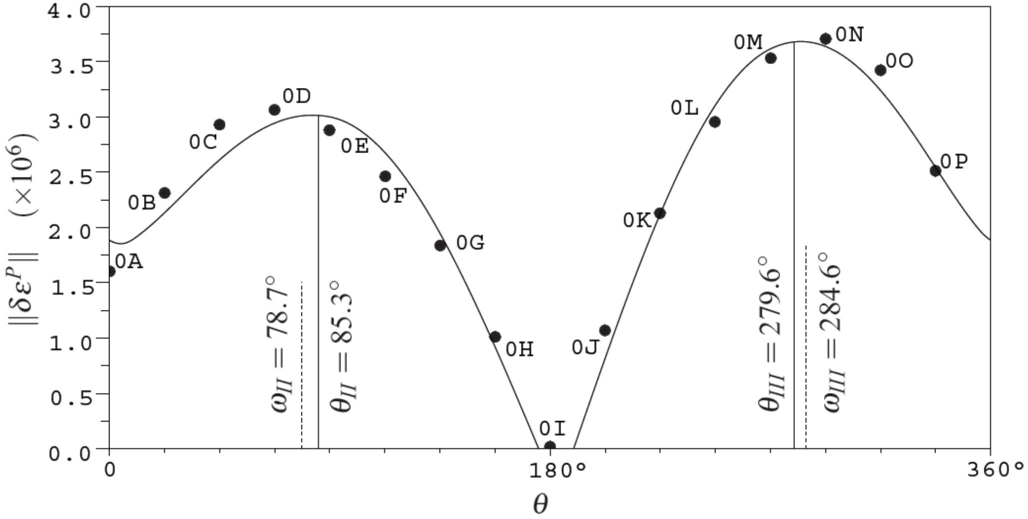
Without rotation of principal stress axes

Stress probing in the plane σ_{11} - σ_{22}



With rotation of principal stress axes

Stress probing involving σ_{12}



- 3 plastic mechanisms of deformation can be identified

- Authors shown that results can be described with an elasto-plastic relation with 3 yield criteria and 3 flow rules.

⇒ Once again the mechanical behaviour is well incrementally non-linear.

$$d\vec{\varepsilon} = M_h\left(\frac{d\vec{\sigma}}{\|d\vec{\sigma}\|}\right) d\vec{\sigma}$$

About directional stress probes and response envelop

Gudehus G., “ A comparison of some constitutive laws for soils under radially symmetric loading and unloading”, *Proc. 3rd Numer. Meth. in Geomechanics*, A. A. Balkema, Aachen, p. 1309-1323, 2-6 April, 1979.

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Froiio F. and Roux J.N., “Incremental response of a model granular material by stress probing with DEM simulations”, *IUTAM-ISIMM Symposium on mathematical modeling and physical instances of regular flow*, Italy, 2009, DOI 10.1063/1.3435388.