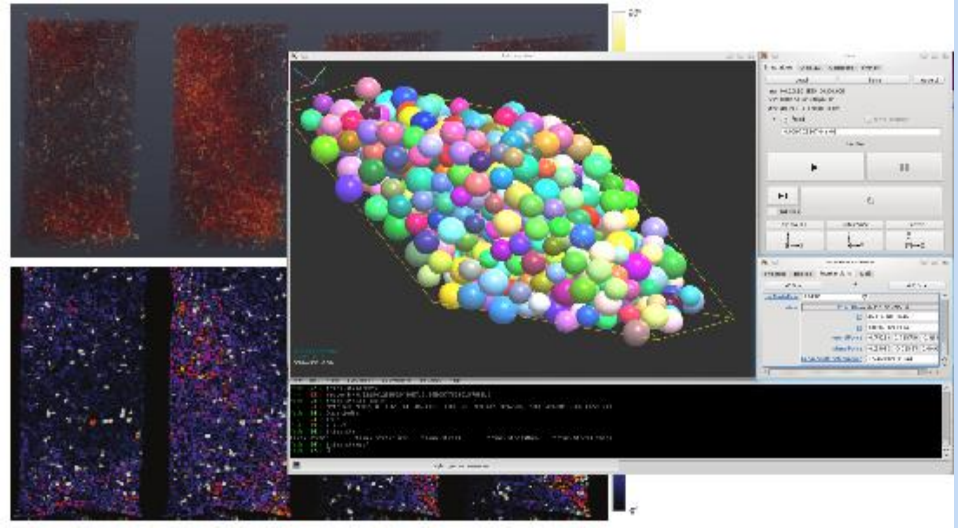

# Discrete Mechanics of Geomaterials
# 3rd Alert Olek Zienkiewicz Course

**June 27th-July 1st**
**3SR, Grenoble, France**

# Homogenisation in granular media : some features

*F. Nicot (CEMAGREF – Grenoble, France)*

# Grains, contacts, REV

A granular medium can be regarded as a **grain assembly** or a **contact network**

Following an homogenization line, a contact description is more appropriated. The contact is the basic constitutive unit of the medium.

Grain deformation is concentrated at contact points, and the macroscopic deformability (on the assembly scale) stems from the relative displacement between grains, involving sliding, rolling and normal compression.

A granular volume is reputed to be a **REV** (Representative Volume Element) when both macro-homogeneous stress and strain fields can be defined.

This requires that all characteristic internal lengths of the medium (grain size, force chain length, etc.) are small with respect to the specimen size.

# Macro-homogeneity and Hill's Lemma

A strain field $\overline{\overline{\varepsilon}}$ is macro-homogeneous within a volume V if

$$\forall\ M\ (\vec{x}) \in \partial V \qquad \vec{u}(M) = \overline{\overline{\varepsilon}}\ \vec{x}$$

with $\overline{\overline{\varepsilon}}$ constant on $\partial V$

Then (**Hill's Lemma**), for any stress tensor $\overline{\overline{\sigma}}$ with zero divergence:

$$\langle \sigma : \varepsilon \rangle = \langle \sigma \rangle : \langle \varepsilon \rangle$$

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A stress field $\overline{\overline{\sigma}}$ with zero divergence is macro-homogeneous within a volume V if

$$\forall\ M\ (\vec{n}) \in \partial V \qquad \vec{F}(M) = \overline{\overline{\sigma}}\ \vec{n}$$

with $\overline{\overline{\sigma}}$ constant on $\partial V$

Then (**Hill's Lemma**), for any strain tensor $\overline{\overline{\varepsilon}}$

$$\langle \sigma : \varepsilon \rangle = \langle \sigma \rangle : \langle \varepsilon \rangle$$

# Macro-homogeneity and Hill's Lemma

**VER**

Fields $\overline{\overline{\varepsilon}}$ and $\overline{\overline{\sigma}}$ are macro-homogeneous

**Hill's lemma**

$$\langle \sigma : \varepsilon \rangle = \langle \sigma \rangle : \langle \varepsilon \rangle$$

**3 main scales :**

- **Microscopic scale**

Contact between two adjoining particles (opening, closure, sliding, liquid bridges, bonds, etc.)

- **Mesoscopic scale**

Intermediate scale corresponding to a discrete set of neighboring particles

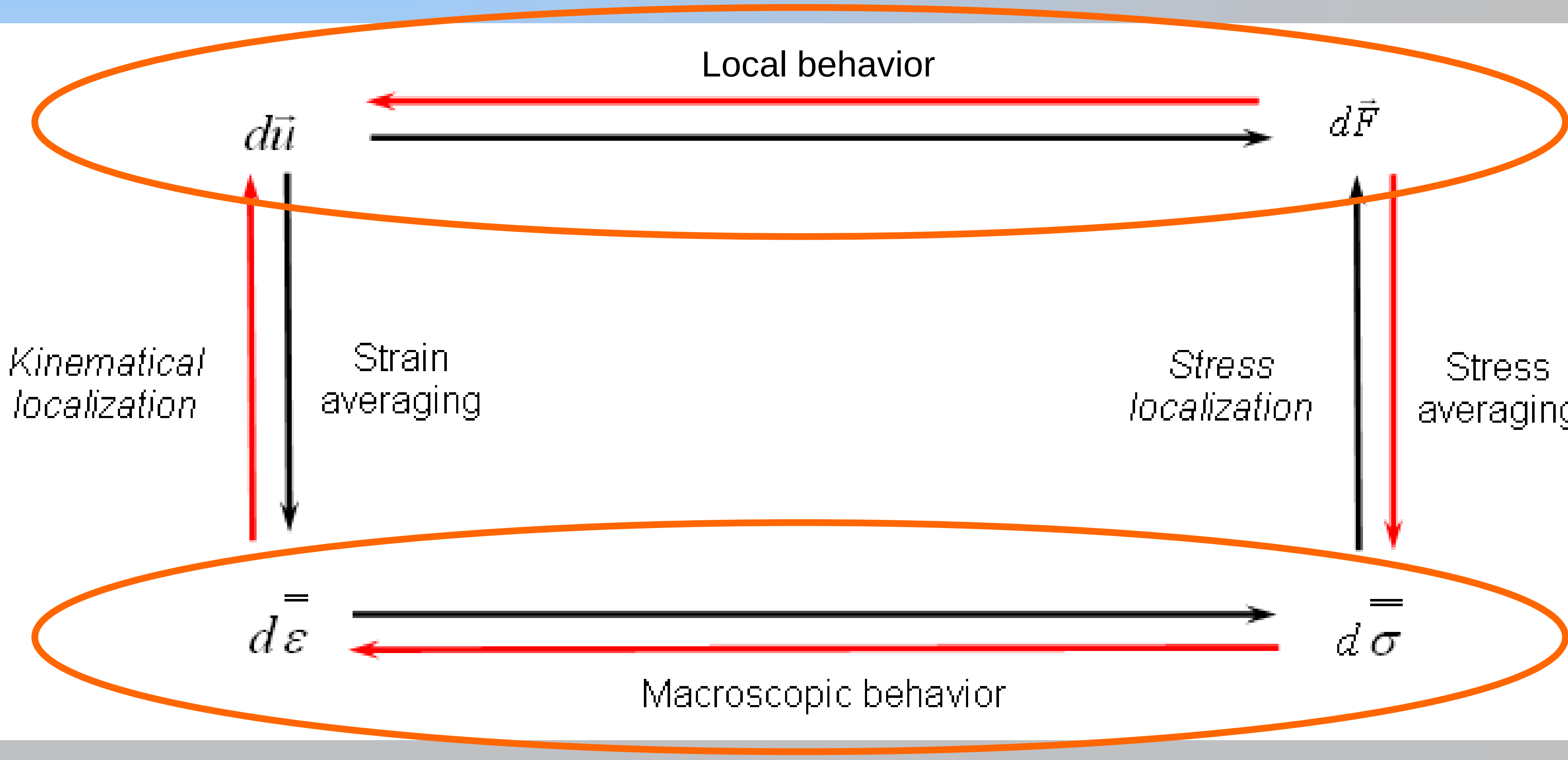
(force chains, undergoing the assembly stability ; Oda, 1972 ; Radjai *et al*., 1998)

- **Macroscopic scale**

Constitutive relations are written on this scale, to be integrated in FEM codes

(boundary value problems)

**MICRO**



Local behavior
$d\vec{u}$
$d\vec{F}$
Kinematical localization
Strain averaging
Stress localization
Stress averaging
$d\bar{\bar{\varepsilon}}$
$d\bar{\bar{\sigma}}$
Macroscopic behavior


**MACRO**

# Tools for homogenization

Ideally, homogenization schemes can be solved if the motion of each particle is described (particulate approach). This is what is done in DEM.

However:

Computations can be time consuming, and very heavy

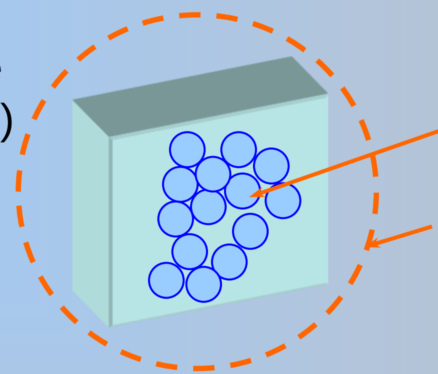
No constitutive equation relating both stress and strain tensors

DEM should be regarded as a numerical experimentation tool, rather than an homogenization technique.

In practice, the balance (or the motion) of each particle is not described:

- The global equilibrium (on the specimen scale) is written (virtual works theorem)
- **Equivalence** between «  macro work » and « sum of micro works »
- A simplified **statistical description** of contact distribution evolution is introduced (heuristic vision of fabrics notion)
- Additional **hypothesis** are required, as the motion of each particle is not considered…

RVE
(Representative
Volume Element)



$\vec{F}, \vec{u}$

$\overline{\overline{\sigma}}, \overline{\overline{\varepsilon}}$

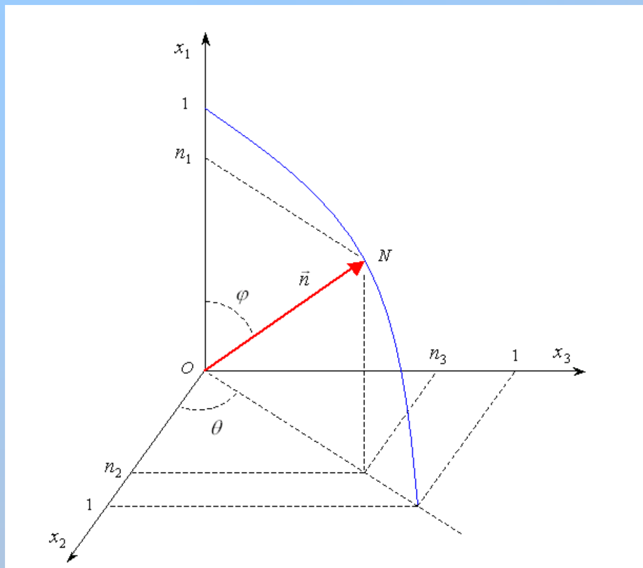
$$d\overline{\overline{\varepsilon}} \longrightarrow d\vec{u} \longrightarrow d\vec{F} \longrightarrow d\overline{\overline{\sigma}}$$

**Strain localization operator**

**Local behavior**

**Stress averaging**

(*Chang*, 1992; *Cambou*, 1993; *Chang* and *Hicher*, 2005; *Nicot* and *Darve*, 2005; *etc*)

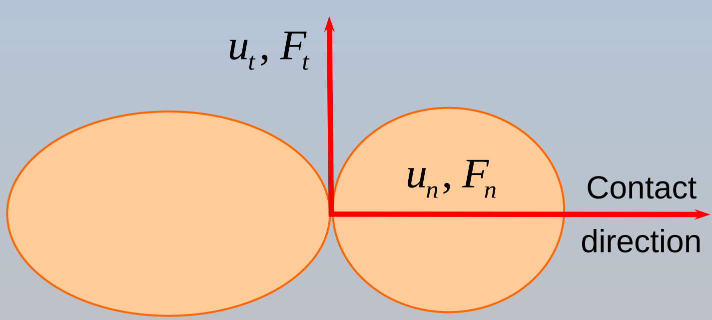



Directional contact vector

$$\vec{n}(\theta,\varphi)=\begin{bmatrix} n_1(\theta,\varphi) \\ n_2(\theta,\varphi) \\ n_3(\theta,\varphi) \end{bmatrix}=\begin{bmatrix} \cos\varphi \\ \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \end{bmatrix}$$

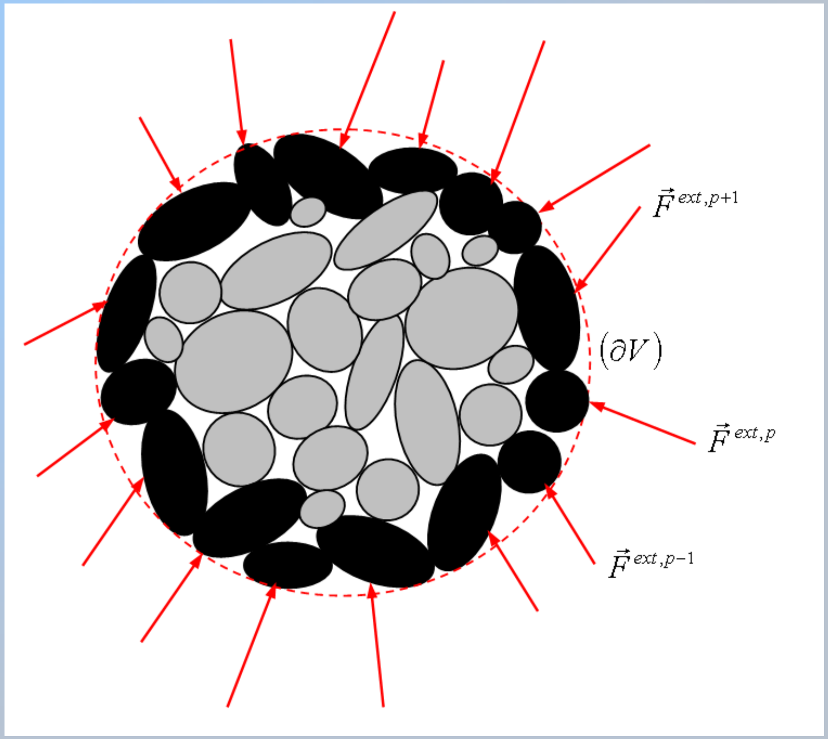
Contact probability

$$f_{\theta,\varphi}(\theta,\varphi)=\frac{\omega_e(\theta,\varphi)}{N_c}$$




$(u_n, u_t)$ Relative displacement of the contact point

$(F_n, F_t)$ Contact forces resulting from the relative displacement




Granular assembly (VER) subjected to a set a external forces applied to boundary particles

Inertial effects are neglected

$$\int_V \sigma_{ij}\ \delta\varepsilon_{ij}\ dv = \sum_{p\in\partial V} F_i^{ext,p}\ \delta u_i^{ext,p}$$

(Virtual work theorem)

$$V\,\overline{\sigma_{ij}\;\delta\varepsilon_{ij}} = \sum_{p\in\partial V} F_i^{ext,p}\;\delta u_i^p$$

$$V\,\overline{\sigma_{ij}\;\delta\varepsilon_{ij}} = V\,\overline{\sigma_{ij}}\;\overline{\delta\varepsilon_{ij}}$$

Hill macro-homogeneity lemma

$$\delta u_i^{ext,p} = -\overline{\delta\varepsilon_{ij}}\;x_j^p$$

$x_j^p$ j^th coordinate of external particle '*p*'

$$\Rightarrow\quad V\,\overline{\sigma_{ij}}\;\overline{\delta\varepsilon_{ij}} = -\sum_{p\in\partial V} F_i^{ext,p}\;\delta x_j^p\;\overline{\delta\varepsilon_{ij}} \qquad \forall\;\overline{\delta\varepsilon_{ij}}$$

$$\Rightarrow\quad \overline{\sigma_{ij}} = -\frac{1}{V}\sum_{p\in\partial V} F_i^{ext,p}\;\delta x_j^p$$

**Boundary Love-Weber formula**

## Discrete formulation

$$F_i^{ext,p} + \sum_{q \in C(p)} F_i^{q,p} = 0$$

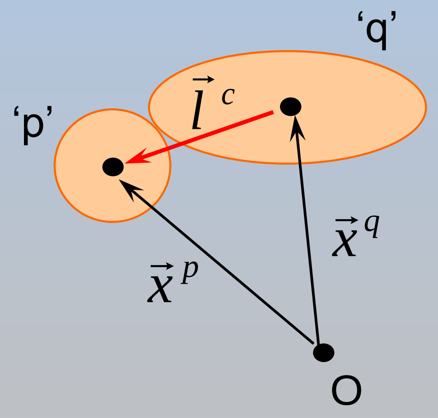
'p' external

C(p) : set of particles 'q' in contact with 'p'

$$\sum_{q \in C(p)} F_i^{q,p} = 0$$

'p' internal

$$\Rightarrow \sum_{p \in \partial V} F_i^{ext,p} \, x_j^p + \sum_{q<p} F_i^{q,p} \left( x_i^p - x_i^q \right) = 0$$



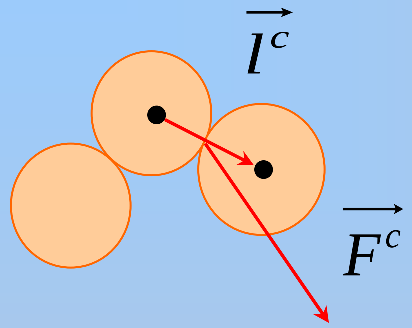

$$\sum_{q<p} F_i^{q,p} \left( x_i^p - x_i^q \right) = \sum_{c=1}^{N_c} F_i^c \, l_j^c$$

$$\Rightarrow \overline{\sigma_{ij}} = \frac{1}{V} \sum_{c=1}^{N_c} F_i^c \, l_j^c$$

**Contact Love-Weber formula**

# Force averaging

## Continuum formulation




on a contact « c »

$$\sigma_{ij} = \frac{1}{v_e} \sum_c F_i^c \, l_j^c$$

(*Love* Formula, 1927)

(*Weber*, 1966; *Mehrabadi*, 1981; *etc*.)

$\hat{F}_i(\vec{n})$ Average contact force along direction $\vec{n}$

Spherical particles, radius $r_g$

$$\sigma_{ij} = \iint_D \frac{2r_g}{v_e} \hat{F}_i(\vec{n})\, n_j \, \omega_e(\vec{n})\, d\Omega$$

Integration over all the contact directions of the physical space

# Behavior on the microscopic scale

**The local behavior accounts for the constitutive specificity of the material:**

- Frictional elasto-plastic model (sands, …)
- Cohesive elasto-plastic model (concrete, …)
- Visco-elasto-plastic behavior (snow, …)
- Adjunction of capillary forces (unsaturated soils)

**Frictional-elastic model**

$$dF_n = k_n \, du_n$$

$$d\vec{F}_t = \min\left\{\left\|\vec{F}_t + k_t \, d\vec{u}_t\right\|, \tan\varphi_g \left(F_n + k_n \, du_n\right)\right\} \frac{\vec{F}_t + k_t \, d\vec{u}_t}{\left\|\vec{F}_t + k_t \, d\vec{u}_t\right\|} - \vec{F}_t$$

$$\begin{bmatrix} dF_n \\ dF_t \end{bmatrix} = \begin{bmatrix} k_n & 0 \\ k_{tn} & k_{tt} \end{bmatrix} \begin{bmatrix} du_n \\ du_t \end{bmatrix}$$

In elastic regime $k_{tt} = k_t$

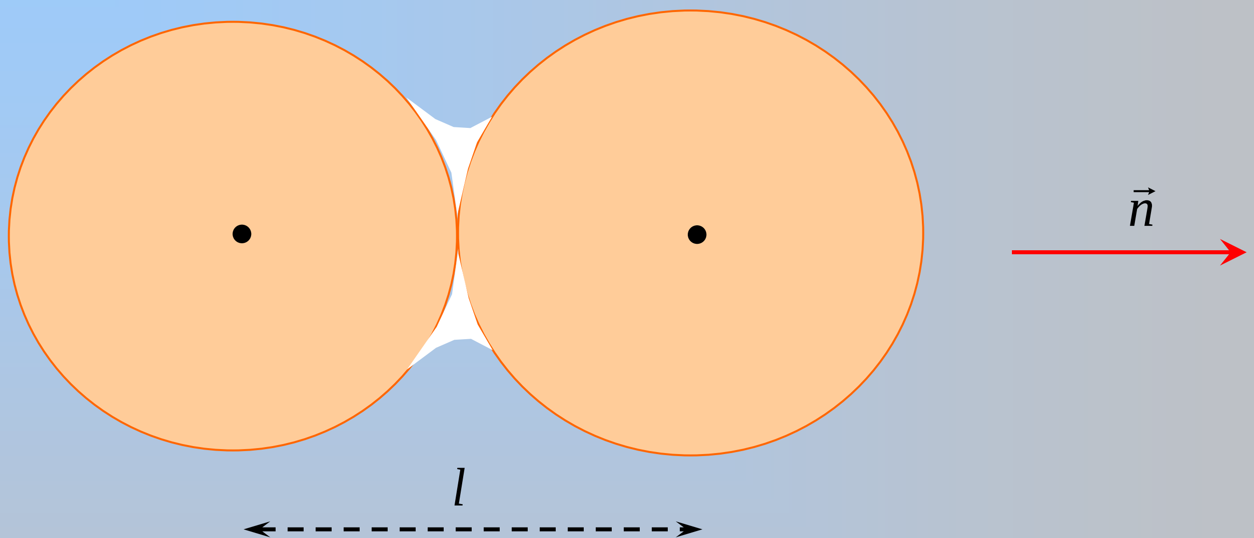
$k_{tn} = 0$

In plastic regime $k_{tt} = 0$

$k_{tn} = \tan\varphi_g \; k_n$

**3 Parameters** : $k_n, k_t, \varphi_g$

Existence of liquid bridges in pendular regime

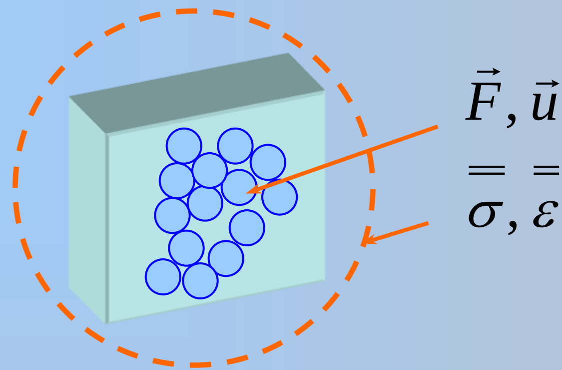


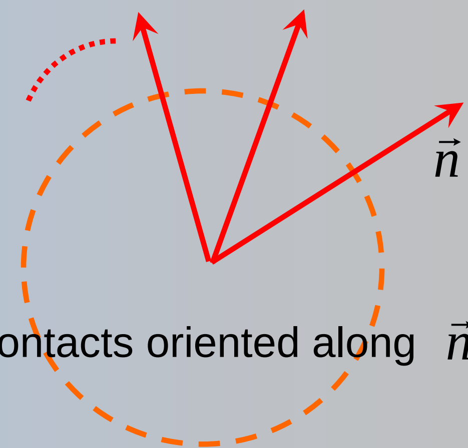

intergranular force $\vec{F}^{g}(\vec{n})$

capillary force $\vec{F}^{cap}(\vec{n})$ Depends on the distance *l*
Obtained from Laplace-Young equation

Soulie *et al*., IJNAMG, 2006; Richefeu *et al*., IJNAMG, 2008
Chang and Hicher, IJSS, 2006; Scholtes *et al*., 2009, IJNAMG

RVE
(Representative
Volume Element)




$N_n$ contacts oriented along $\vec{n}$

Using local variables

$$\dot{e}_d = \sum_{p=1}^{N} \sum_{q=1}^{p-1} \left( \vec{F}^{\,p,q} \cdot \dot{\vec{u}}^{\,p,q} \right) = \sum_{\vec{n}} \sum_{c_n=1}^{N_n} \left( F_i^{c_n} \, \dot{u}_i^{c_n} \right)$$

$$\sum_{\vec{n}} N_n = N_c$$

Using macroscopic variables

$$\dot{E}_d = V \, \sigma_{ij} \, \dot{\varepsilon}_{ij}$$

$$\dot{e}_d = \dot{E}_d$$

$$V\,\sigma_{ij} = 2r_g \sum_{\vec{n}} \left( \sum_{c_n=1}^{N_n} F_i^{c_n} \right) n_j$$

Love-Weber

Along each direction *n* :

Average force $\hat{F}_i$

$$N_n\,\hat{F}_i = \sum_{c_n=1}^{N_n} \left( F_i^{c_n} \right)$$

Kinematic variable $\hat{u}_i$

$$N_n\,\hat{F}_i\,\dot{\hat{u}}_i = \sum_{c_n=1}^{N_n} \left( F_i^{c_n}\,\dot{u}_i^{c_n} \right)$$

!

$$\frac{1}{N_n} \sum_{c_n=1}^{N_n} \left( F_i^{c_n}\,\dot{u}_i^{c_n} \right) \neq \left( \frac{1}{N_n} \sum_{c_n=1}^{N_n} F_i^{c_n} \right) \left( \frac{1}{N_n} \sum_{c_n=1}^{N_n} \dot{u}_i^{c_n} \right)$$

$$\dot{\hat{u}}_i \neq \frac{1}{N_n} \sum_{c_n=1}^{N_n} \dot{u}_i^{c_n}$$

$$V\,\sigma_{ij} = 2r_g \sum_{\vec{n}} \left( \sum_{c_n=1}^{N_n} F_i^{c_n} \right) n_j = 2r_g\; N_n \sum_{\vec{n}} \hat{F}_i\; n_j$$

$$\dot{e}_d = \dot{E}_d \quad \Rightarrow \quad V\,\sigma_{ij}\;\dot{\varepsilon}_{ij} = \sum_{\vec{n}} \sum_{c_n=1}^{N_n} \left( F_i^{c_n}\; \dot{u}_i^{c_n} \right)$$

$$N_n\; \hat{F}_i\; \dot{\hat{u}}_i = \sum_{c_n=1}^{N_n} \left( F_i^{c_n}\; \dot{u}_i^{c_n} \right)$$

$$\left.\right\} \; V\,\sigma_{ij}\;\dot{\varepsilon}_{ij} = N_n \sum_{\vec{n}} \hat{F}_i\; \dot{\hat{u}}_i$$

$$\Rightarrow \quad \sum_{\vec{n}} \left( \hat{F}_i \left( \dot{\hat{u}}_i - 2r_g\; \dot{\varepsilon}_{ij}\; n_j \right) \right) = 0$$

$$\sum_{\vec{n}} \left( \hat{F}_i \left( \dot{\hat{u}}_i - 2r_g \, \dot{\varepsilon}_{ij} \, n_j \right) \right) = 0$$

Discrete formulation

$$\int_{\Omega} \hat{F}_i \left( \dot{\hat{u}}_i - 2r_g \, \dot{\varepsilon}_{ij} \, n_j \right) \omega \, d\Omega = 0$$

Continuous formulation

⇒ **3 reasons preventing the simple relation**

$$\dot{\hat{u}}_i(\vec{n}) = 2r_g \, \dot{\varepsilon}_{ij} \, n_j$$

**1** $\int_{\Omega} \hat{F}_i \left( \dot{\hat{u}}_i - 2r_g \, \dot{\varepsilon}_{ij} \, n_j \right) \omega \, d\Omega = 0 \quad \not\Leftrightarrow \quad \sum_{i} \hat{F}_i \left( \dot{\hat{u}}_i - 2r_g \, \dot{\varepsilon}_{ij} \, n_j \right) = 0$

**2** $\sum_{i} \hat{F}_i \left( \dot{\hat{u}}_i - 2r_g \, \dot{\varepsilon}_{ij} \, n_j \right) = 0 \quad \not\Leftrightarrow \quad \forall i \quad \dot{\hat{u}}_i - 2r_g \, \dot{\varepsilon}_{ij} \, n_j = 0$

**3** $N_n \, \dot{\hat{u}}_i \neq \sum_{c_n=1}^{N_n} \left( \dot{u}_i^{c_n} \right)$ as $\left\langle F_i \, \dot{u}_i \right\rangle_{\vec{n}} \neq \left\langle F_i \right\rangle_{\vec{n}} \left\langle \dot{u}_i \right\rangle_{\vec{n}}$ along each direction *n*

So what ?...

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ComGeo II

# Multiscale approach for granular materials including an intermediate scale

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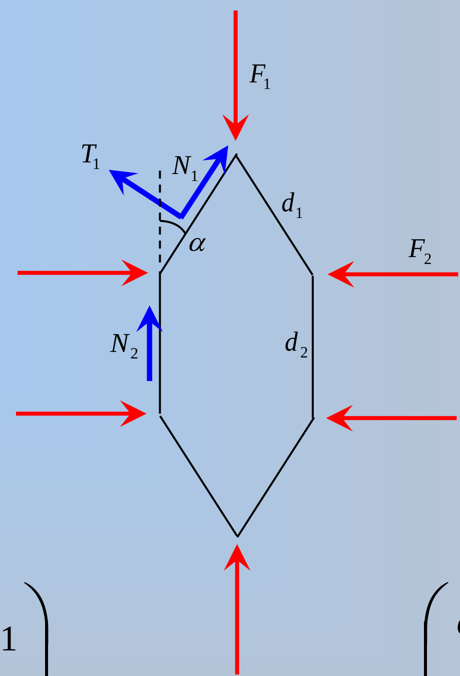
# The *H*-microdirectional model

*2D modeling*

## Homogenization scheme



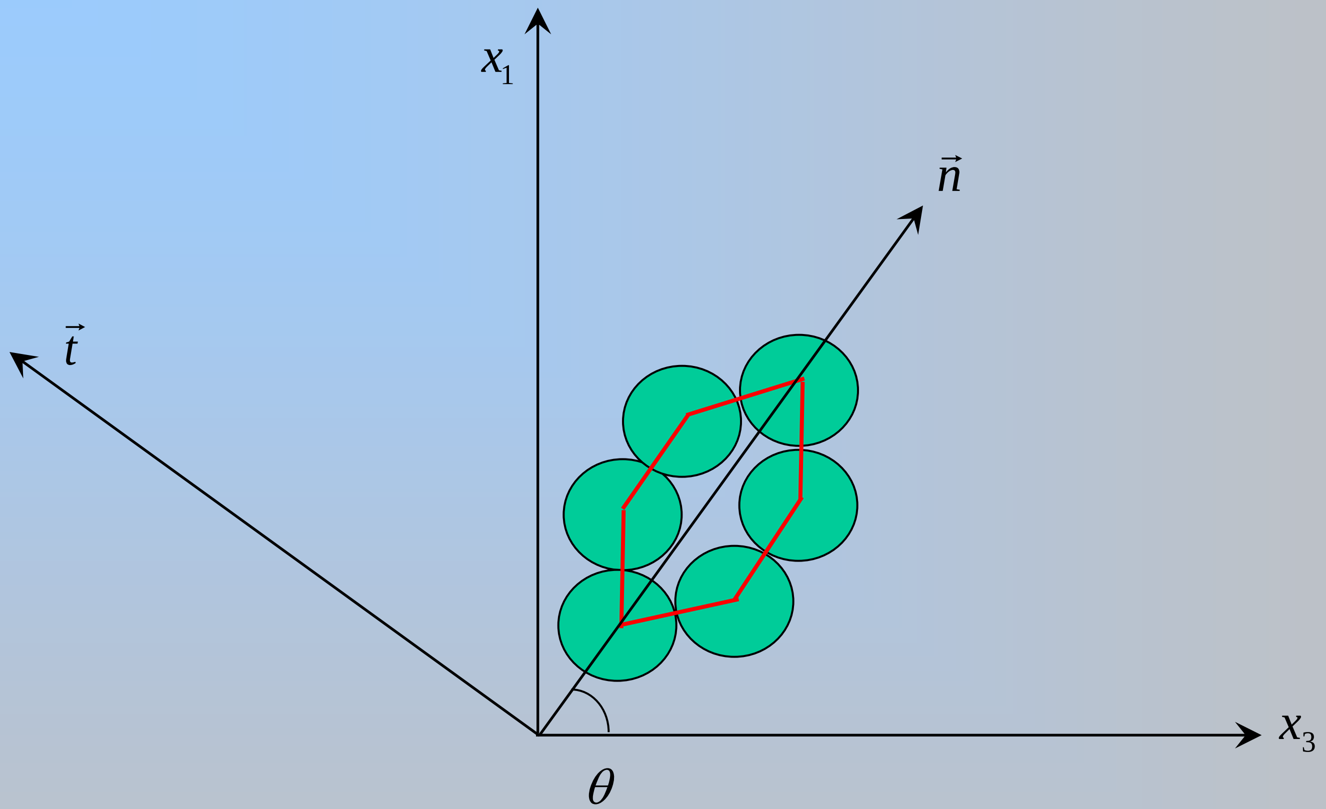

$$\delta\overline{\overline{\varepsilon}} \xrightarrow{\text{Strain localization operator}} \begin{pmatrix} \delta d_1 \\ \delta d_1 \\ \delta \alpha \end{pmatrix} \xrightarrow{\text{Local behavior}} \begin{pmatrix} \delta N_1 \\ \delta T_1 \\ \delta N_2 \end{pmatrix} \xrightarrow{\text{Stress averaging}} d\overline{\overline{\sigma}}$$

**Strain localization operator**

**Local behavior**

**Stress averaging**

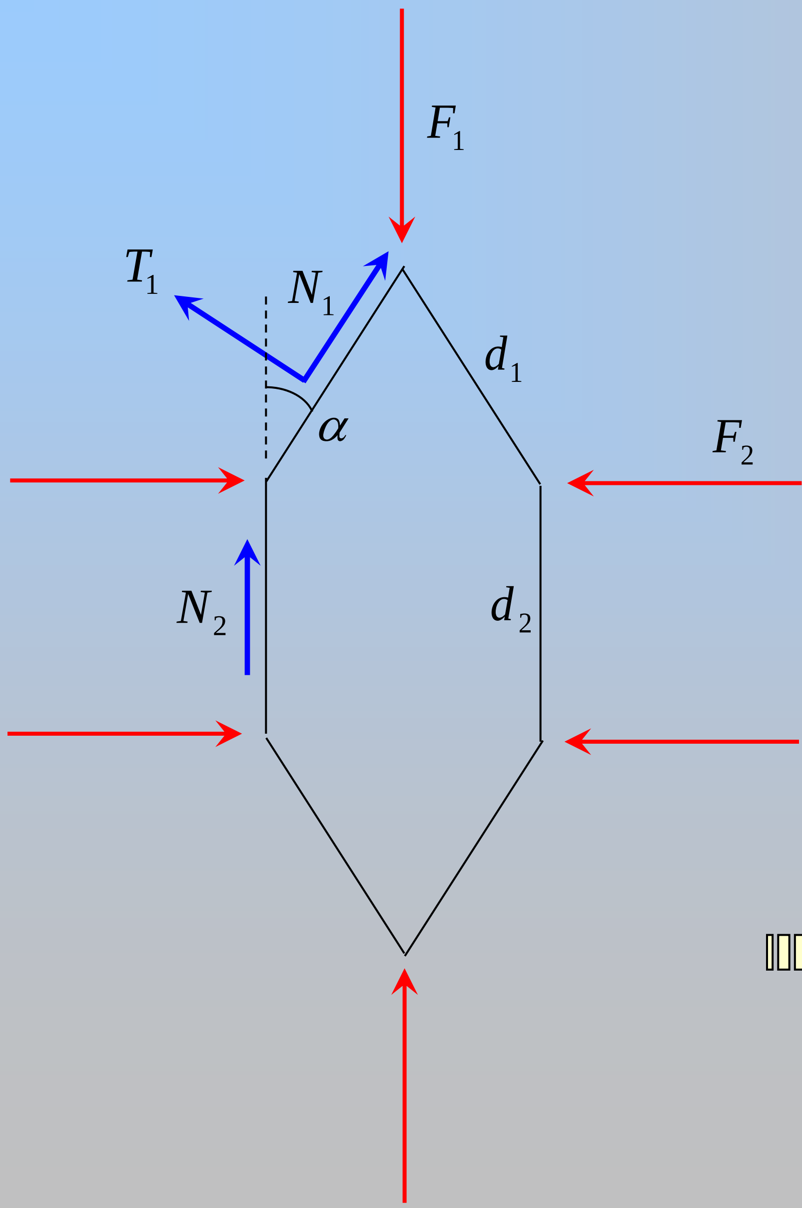
# Introduction of an intermediate scale




Spatial distribution of hexagonal patterns, symmetric with respect to the orientation direction $\vec{n}$

Grains are spherical with the same radius

# Computing contact forces




$$\delta N_1 = -k_n \, \delta d_1$$

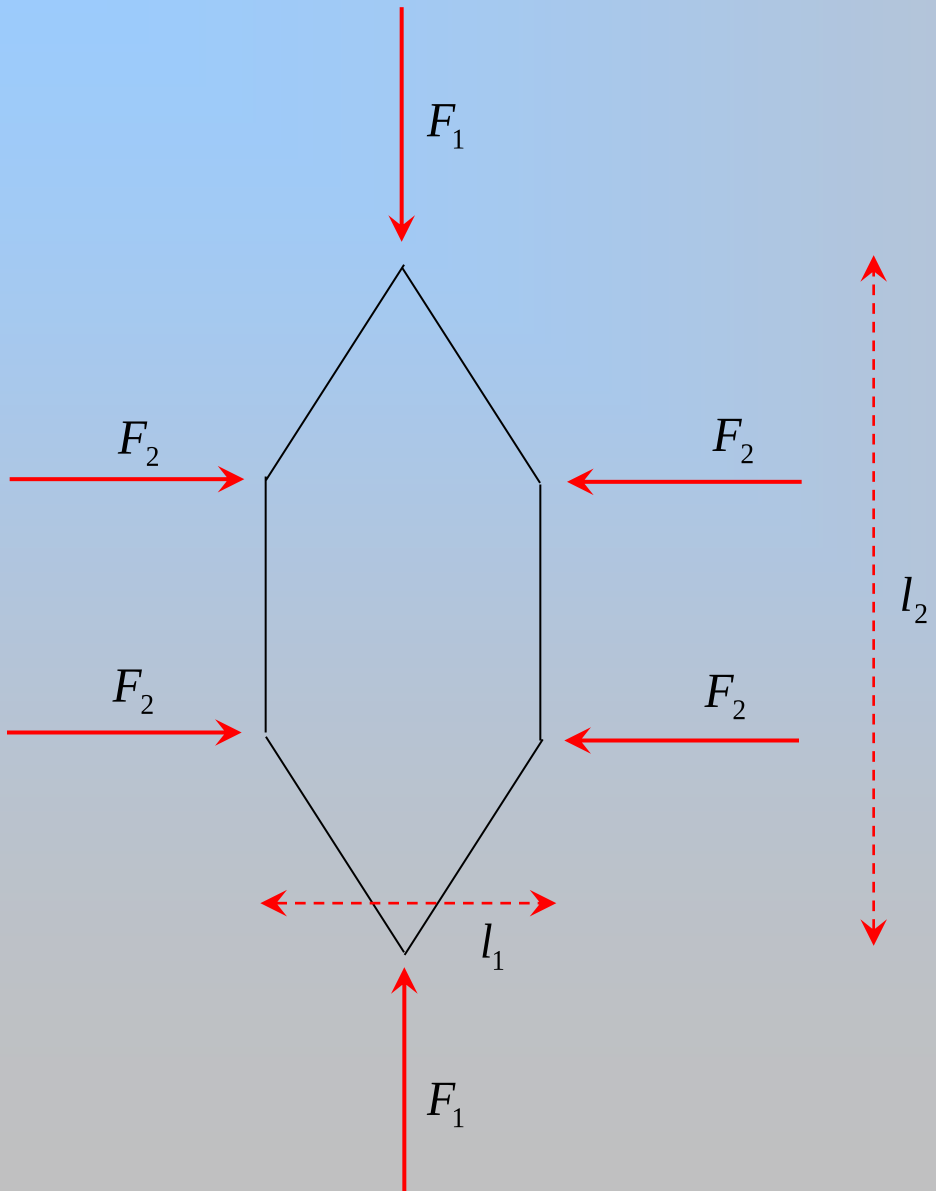
$$\delta T_1 = -k_t \, d_1 \, \delta\alpha$$

$$|T_1| \leq \tan\varphi_g \, N_1$$

$$\delta N_2 = -k_n \, \delta d_2$$

$$\Rightarrow \quad (N_1, T_1, N_2) = f(d_1, d_2, \alpha)$$

# Fundamental kinematic assumption




$$\delta l_1 = -l_1 \, \delta \underline{\underline{\varepsilon}} \, n \, n$$

$$\delta l_2 = -l_2 \, \delta \underline{\underline{\varepsilon}} \, t \, t$$

Elastic regime

$$\begin{bmatrix} 2\cos\alpha & 1 & -2d_1\sin\alpha \\ 2\sin\alpha & 0 & 2d_1\cos\alpha \\ \cos\alpha & -1 & \dfrac{\left(k_t\ d_1+N_1\right)\sin\alpha - T_1\cos\alpha}{k_n} \end{bmatrix} \begin{bmatrix} \delta d_1 \\ \delta d_2 \\ \delta\alpha \end{bmatrix} = \begin{bmatrix} \delta l_1 \\ \delta l_2 \\ 0 \end{bmatrix}$$

Elasto-plastic regime

$$\begin{bmatrix} 2\cos\alpha & 1 & -2d_1\sin\alpha \\ 2\sin\alpha & 0 & 2d_1\cos\alpha \\ \cos\alpha & -1 & \dfrac{N_1\sin\alpha - T_1\cos\alpha}{k_n} \end{bmatrix} \begin{bmatrix} \delta d_1 \\ \delta d_2 \\ \delta\alpha \end{bmatrix} = \begin{bmatrix} \delta l_1 \\ \delta l_2 \\ -\dfrac{\delta T_1\sin\alpha}{k_n} \end{bmatrix}$$

$$\left(\delta d_1, \delta d_2, \delta\alpha\right) = f\left(\delta \overline{\overline{\varepsilon}}\right)$$

# Computing stress tensors
# Both on the meso and macro scales

For the hexagonal pattern

$$\overline{\overline{\sigma}}(\vec{n}) = \begin{bmatrix} \tilde{\sigma}_1 & 0 \\ 0 & \tilde{\sigma}_2 \end{bmatrix}$$

$$V(\vec{n})\,\tilde{\sigma}_1 = 4N_1\,d_1\cos^2\alpha - 4T_1\,d_1\cos\alpha\sin\alpha + 2N_2\,d_2$$

$$V(\vec{n})\,\tilde{\sigma}_2 = 4N_1\,d_1\sin^2\alpha + 4T_1\,d_1\cos\alpha\sin\alpha$$

For the VER

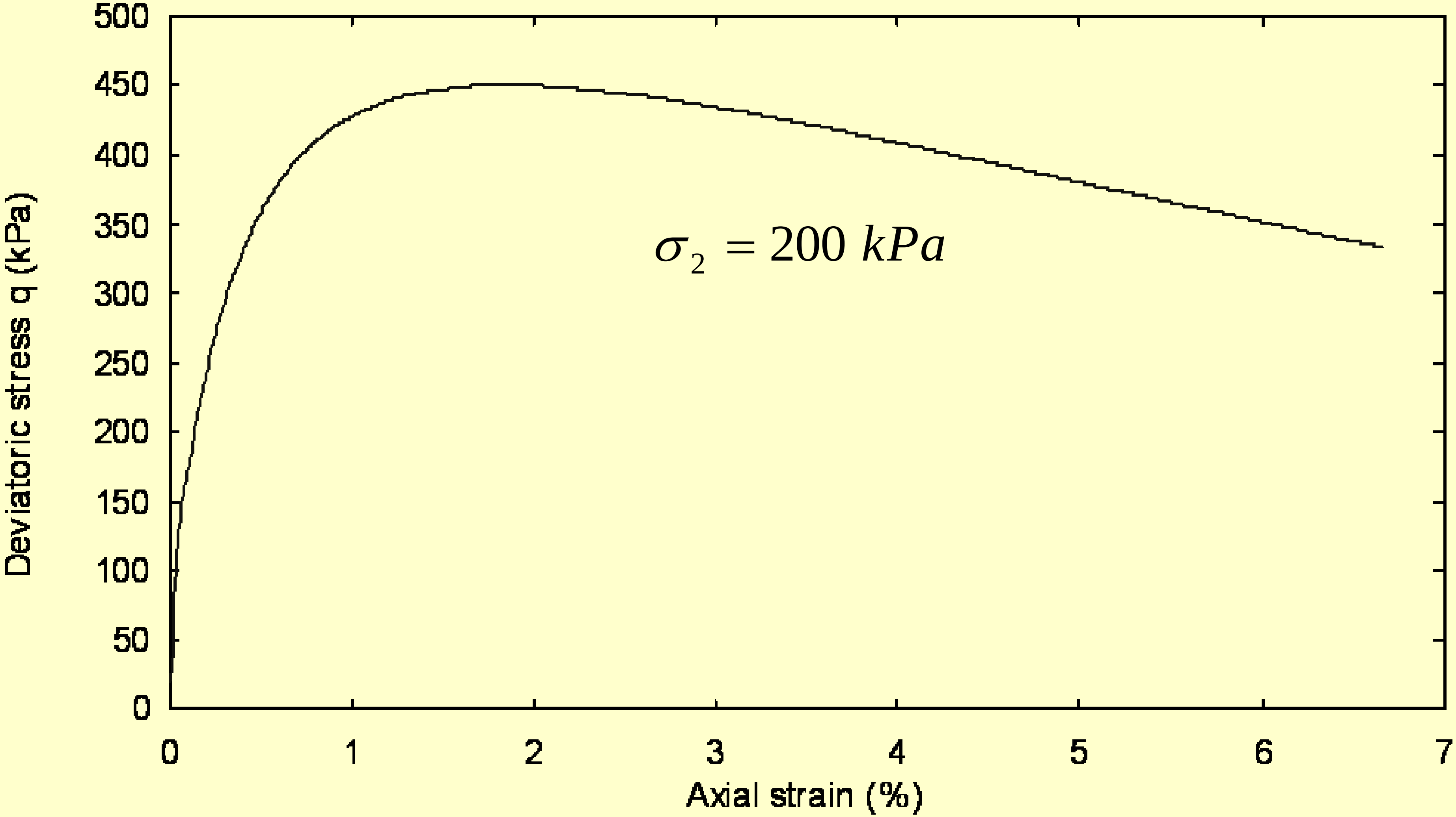
$$\overline{\overline{\sigma}} = \frac{1}{V}\int \omega_e(\vec{n})\,\overline{\overline{P}}^{-1}\begin{bmatrix} V(\vec{n})\,\tilde{\sigma}_1 & 0 \\ 0 & V(\vec{n})\,\tilde{\sigma}_2 \end{bmatrix}\overline{\overline{P}}\,d\theta$$

$$\overline{\overline{P}} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Based on the Love-Weber stress averaging

## Drained biaxial test

| | | | |
|---|---|---|---|
| $k_n = 1000$ | kN/m | $\varphi_g = 20\ \text{deg}$ | |
| $k_t = 500$ | kN/m | $\alpha_o = 42\ \text{deg}$ | |



| | | | |
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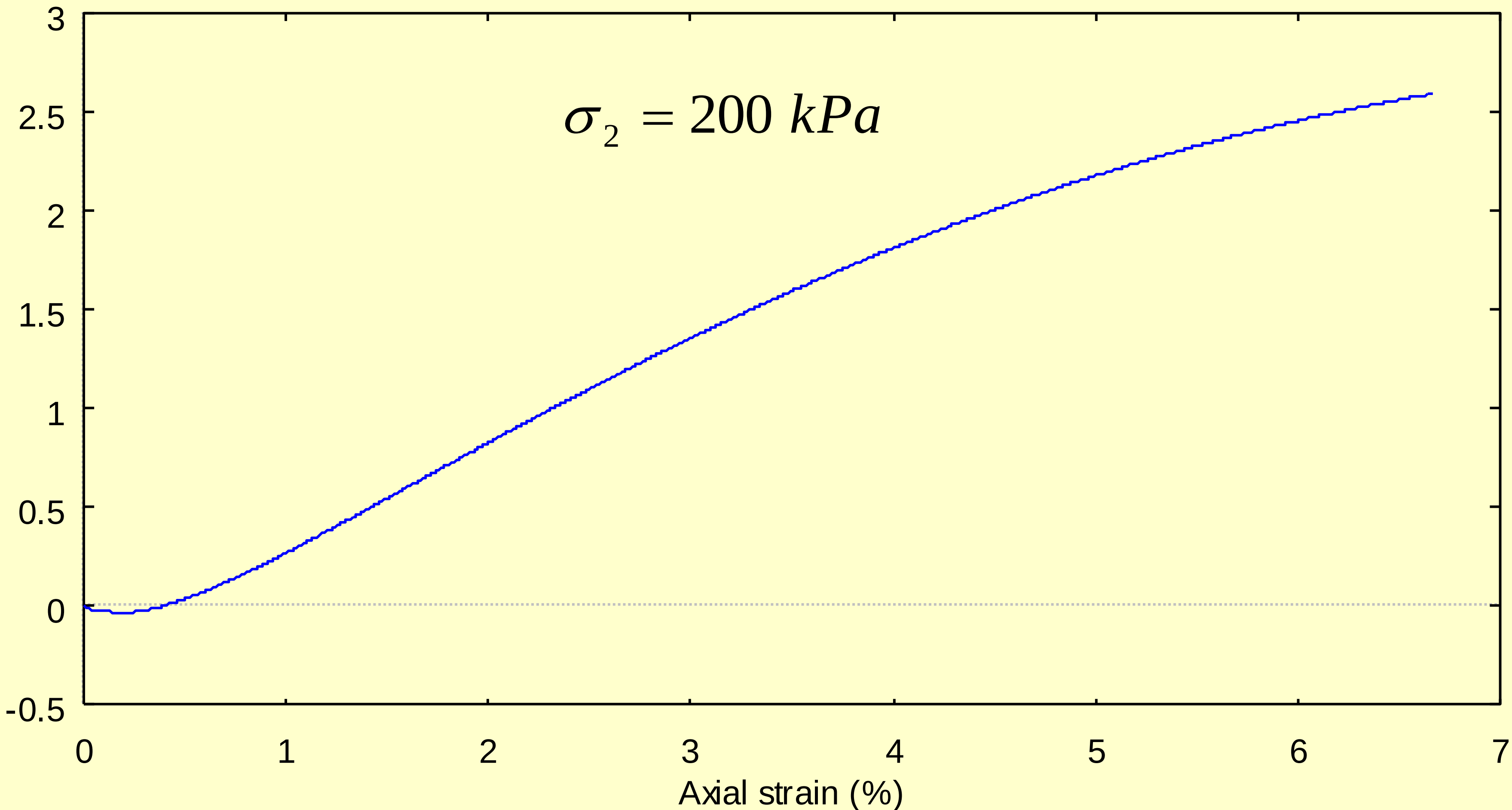
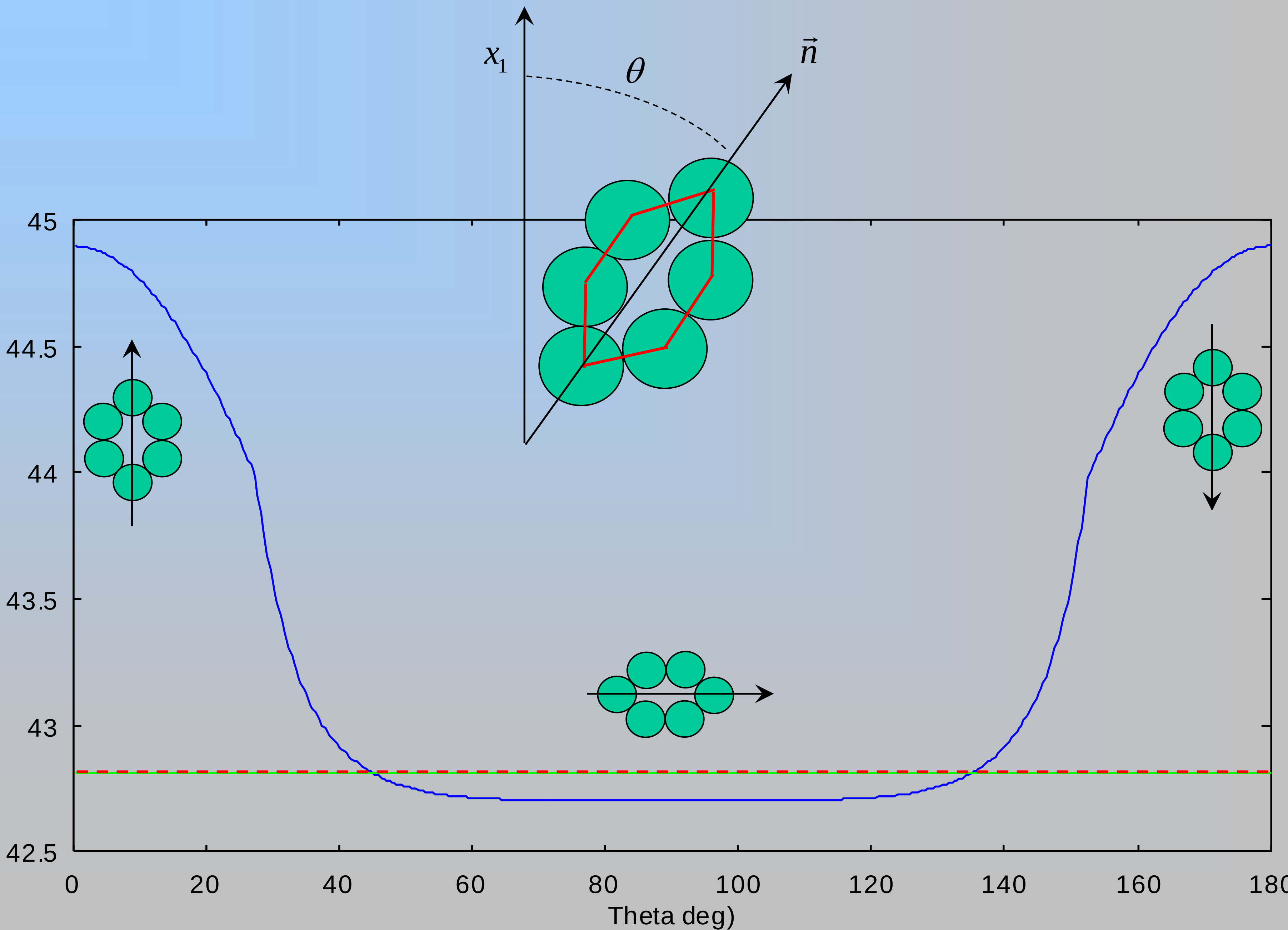


# Drained biaxial test



# Proportional strain loading paths

Loading conditions

$$\begin{cases} \delta\varepsilon_1 + 2R\delta\varepsilon_3 = 0 \\ \delta\varepsilon_1 = const \end{cases}$$

for different R values

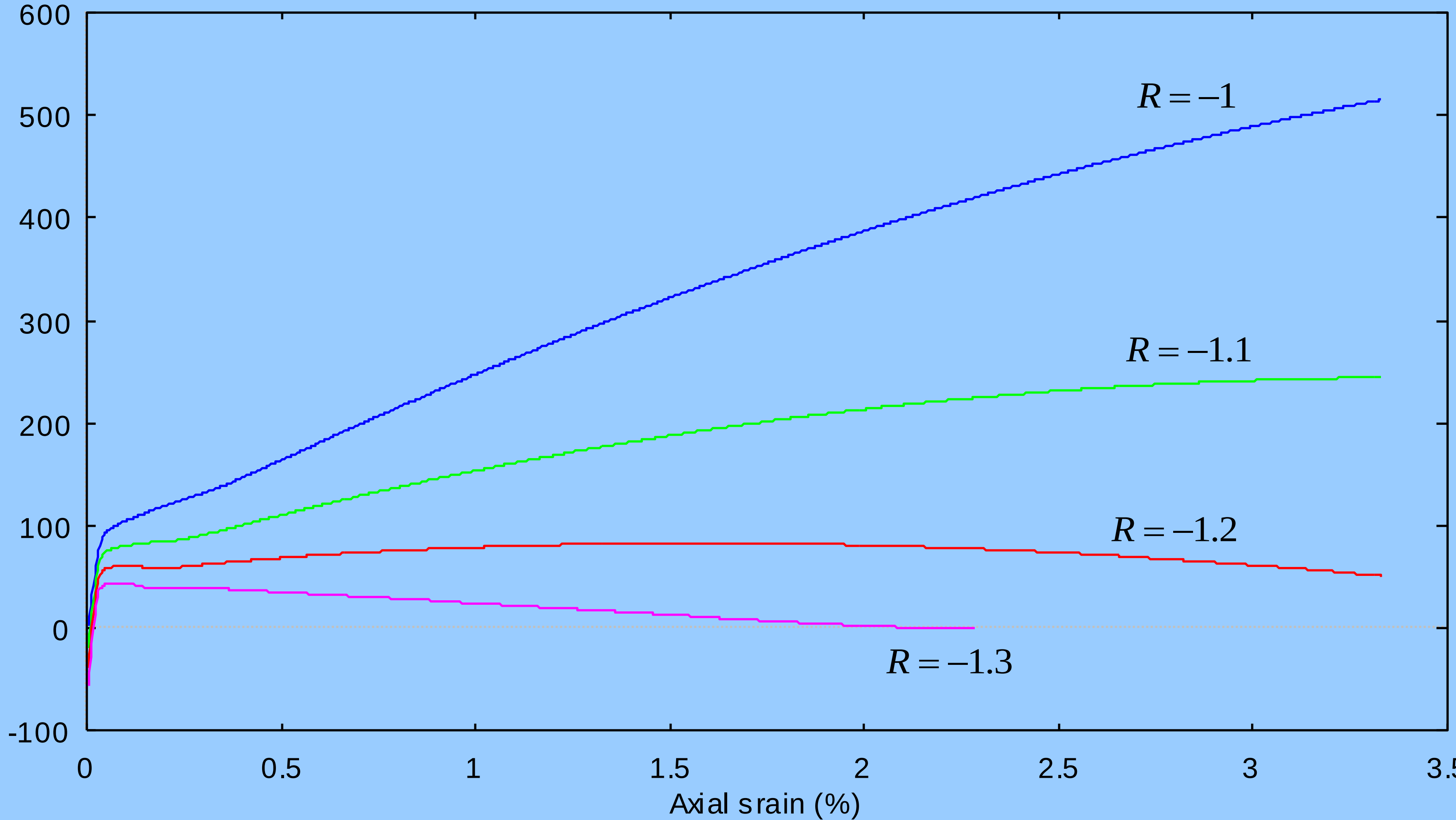
Response analysis

$$\sigma_1 - \frac{\sigma_3}{R}$$ versus $$\varepsilon_1$$

# Proportional strain paths

$$\delta\varepsilon_1 = const$$
$$\delta\varepsilon_2 = R\ \delta\varepsilon_1$$

$$k_n = 1000 \quad \text{kN/m} \qquad \varphi_g = 20\ \text{deg}$$
$$k_t = 500 \quad \text{kN/m} \qquad \alpha_o = 49\ \text{deg}$$



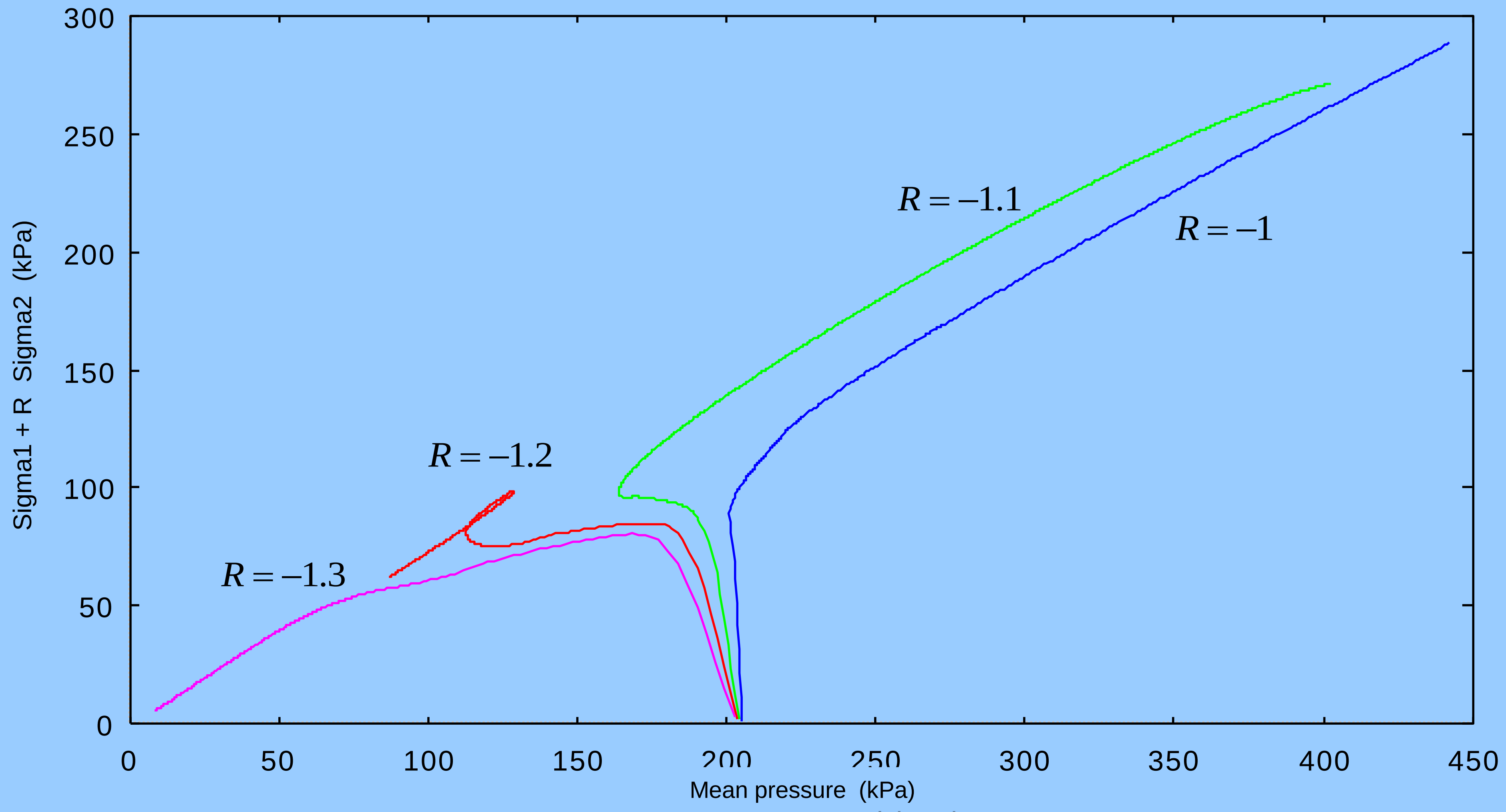
# Proportional strain paths

$\delta\varepsilon_1 = const$

$\delta\varepsilon_2 = R\ \delta\varepsilon_1$

$k_n = 1000$ kN/m $\quad \varphi_g = 20\ \text{deg}$

$k_t = 500$ kN/m $\quad \alpha_o = 49\ \text{deg}$



# Conclusion

Accounting for a mesoscopic scale appears to be a convenient way to overcome the kinematic localization procedure

The H-microdirectional model requires only three constitutive parameters to be identified

Very good qualitative agreement along current loading paths

The macroscopic complexity (richness) of the response is due to the spatial distribution of the hexagons in a variety of mechanical states, not to a local constitutive refinement

Makes it possible to retrieve some of the constitutive features of granular assemblies.

The model can be easily implemented within a FEM code to solve BVP