Homogenization from the viewpoint of the periodic media
Principles and contributions

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What is meant by homogenization ? or upscalling or micro macro methods

To define for a **finely heterogeneous** medium, ”macroscopic” representative quantities and equations that enable to determine them

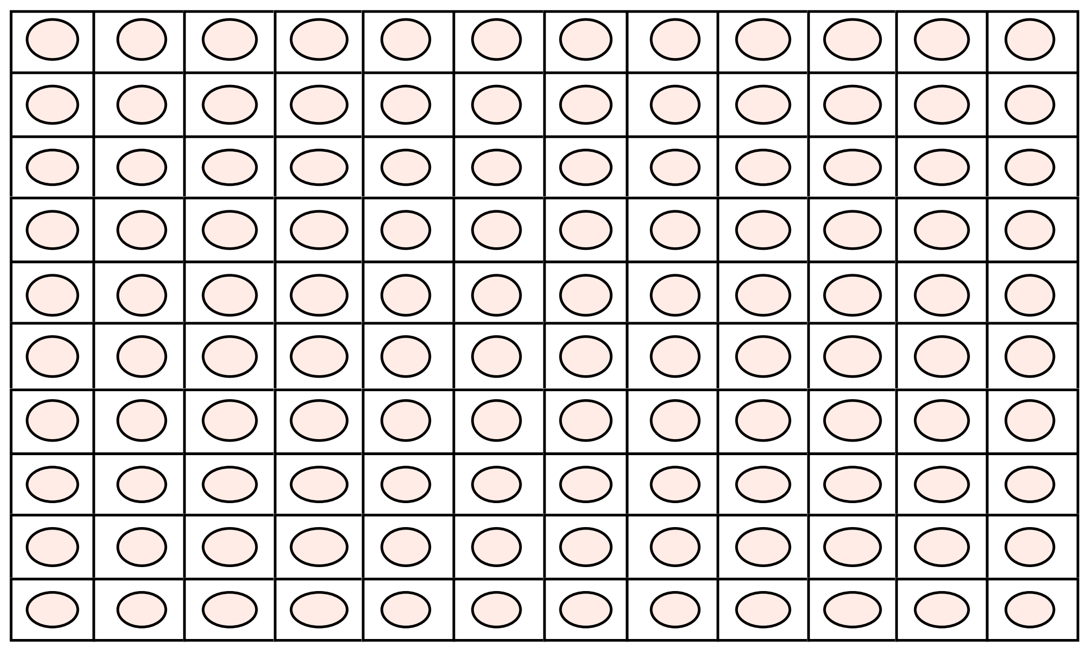
The keyword is **separation of scales**

Contents

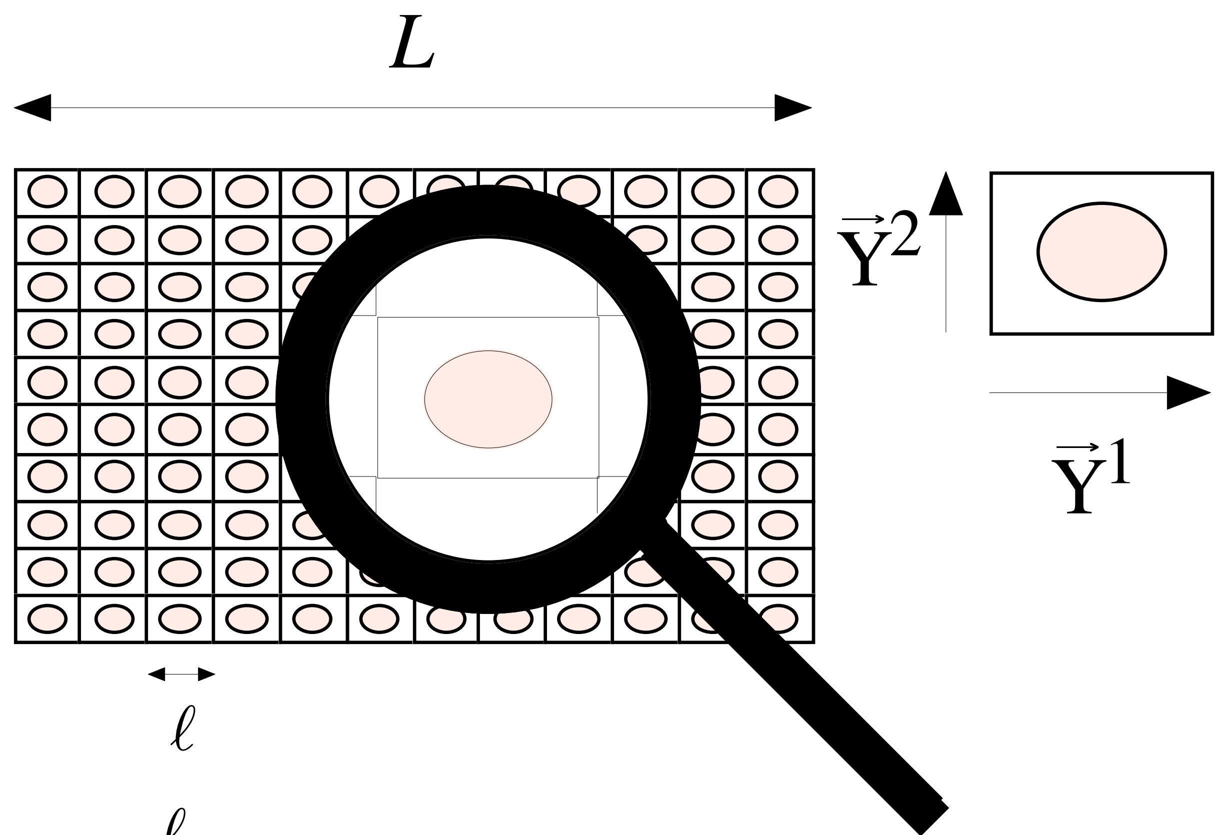
1. Periodic media - Why?
2. 1D bar
3. Double scale asymptotic expansion
4. Self equilibrium problem on the cell of periodicity
5. Multiphysics
6. Related situations - quasi periodic media: large strain, elastoplasticity, ...
7. Intermediate conclusions and remarks
8. Parallel with experiments - VER
9. Granular materials (time permitted)

Periodic media

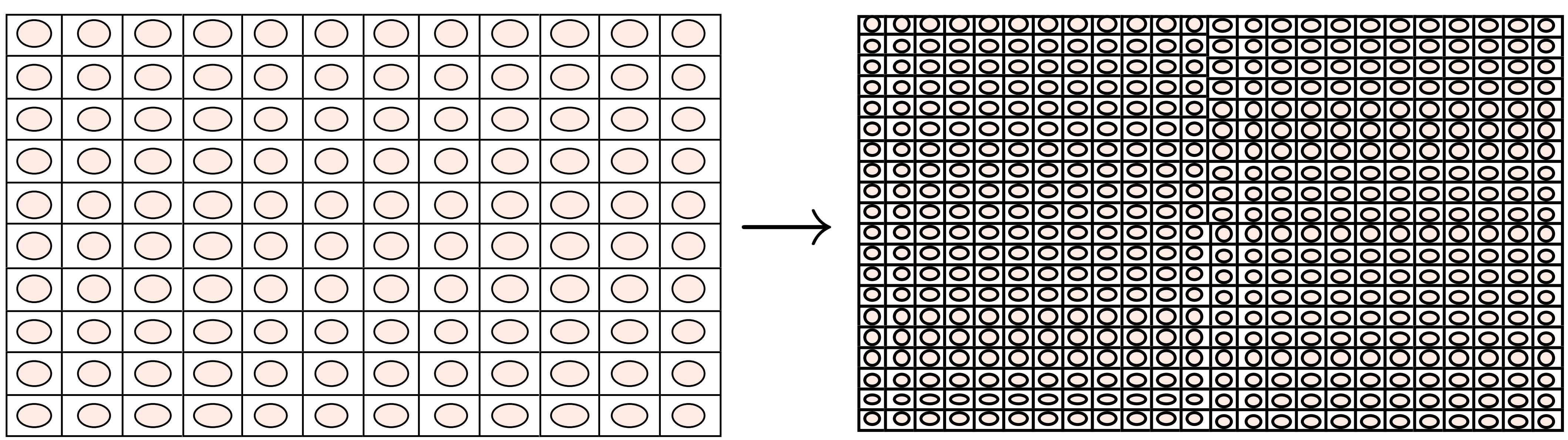
Media with periodically distributed heterogeneities



Finely periodic media - asymptotic method



$$e = \frac{\ell}{L} \to 0$$

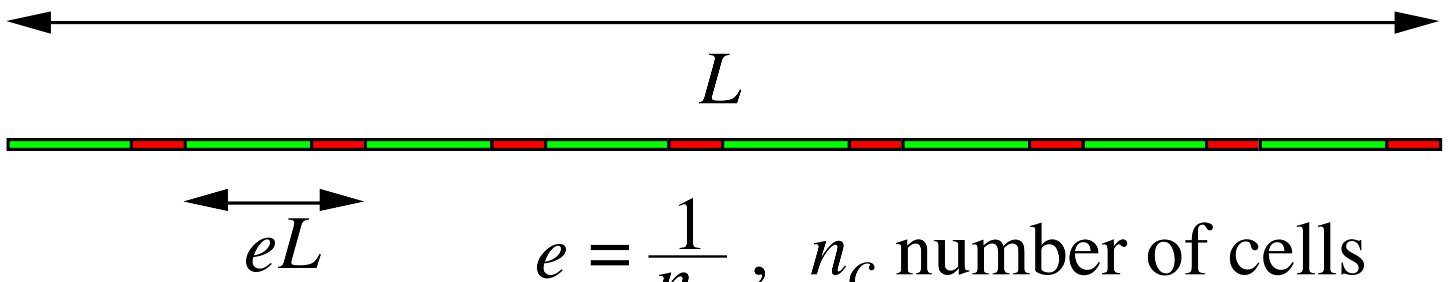


Periodic media ? Why ?

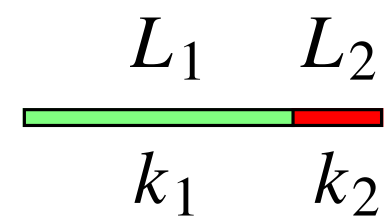
Because there are natural or manufactured periodic media

Mainly because the method which has been widely used is very robust, it has many interesting features and can bring interesting persepectives to homogenization in general

1D periodic bar



$e = \dfrac{1}{n_c}$, n_c number of cells



$$0 = \frac{dN}{dx} + f$$

$$N = k^e(x)\,\frac{du}{dx}$$

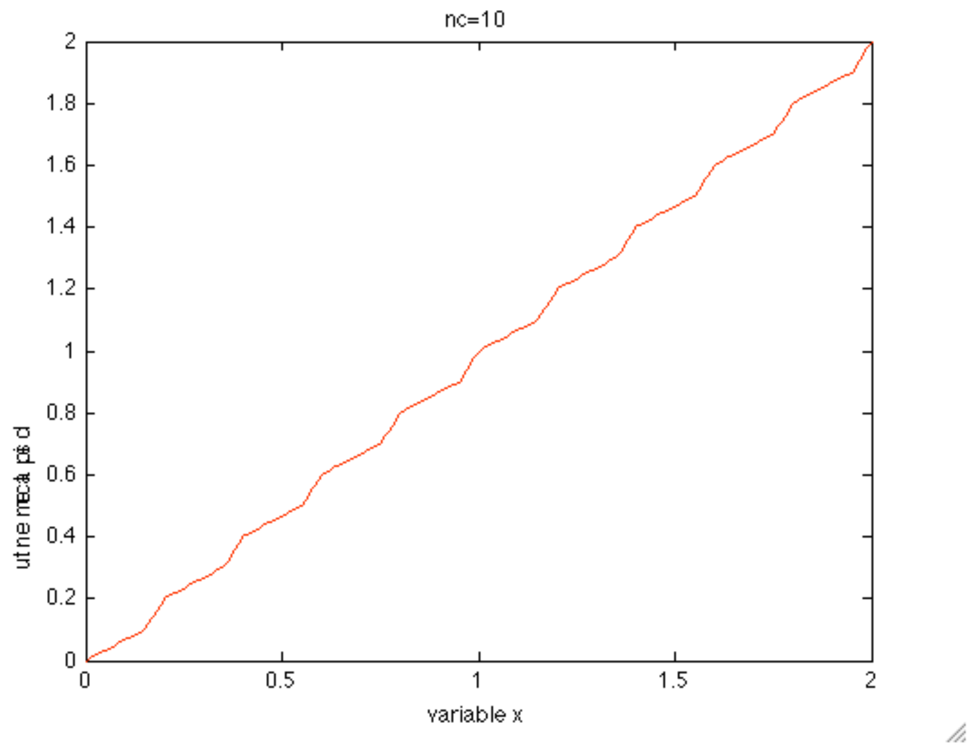
$$k^e(x) = k\left(\frac{x}{e}\right) \, , \, k(y) = \begin{cases} k_1 & 0 < y < L_1 \\ k_2 & L_1 < y < L \end{cases}$$

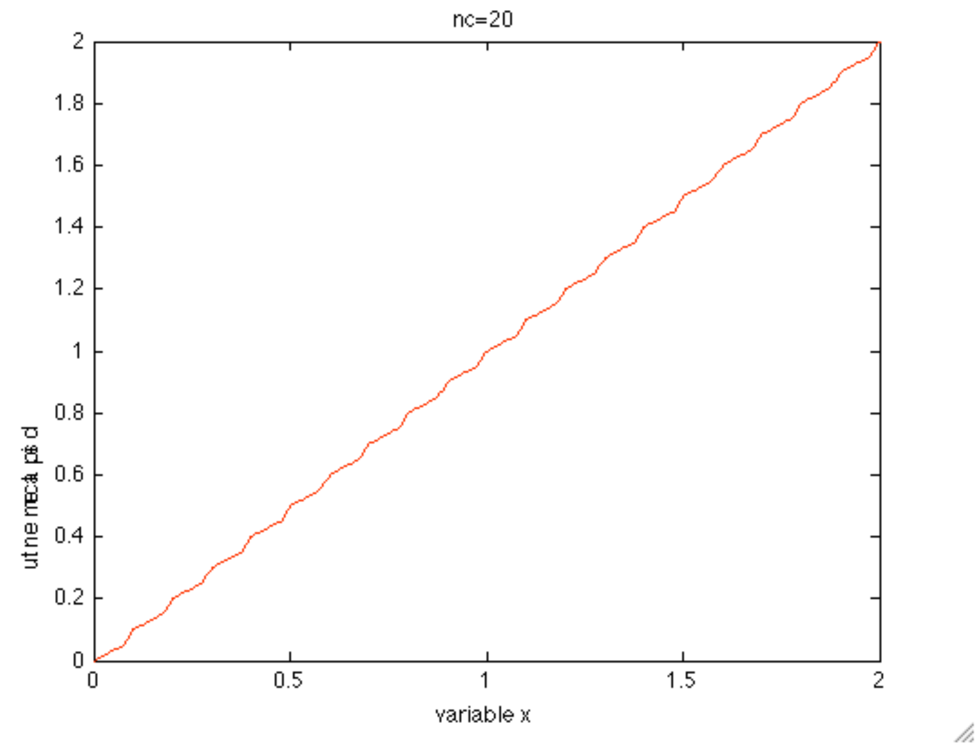
$$u(0) = 0 \, , \; N(L) = F$$

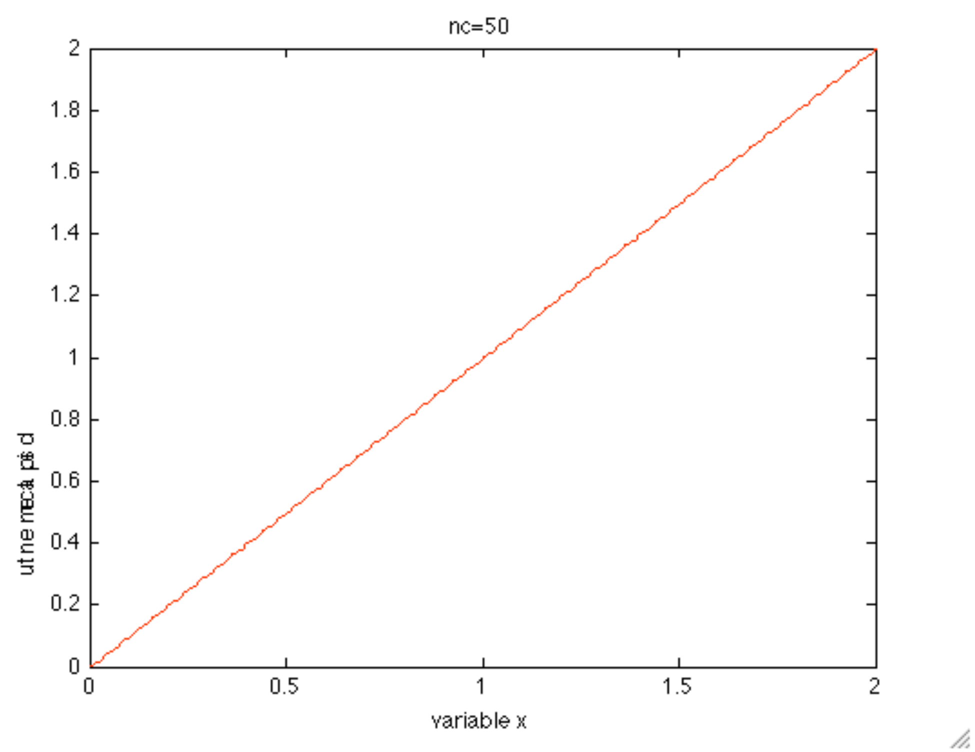
This problem can be easily solved

1D periodic bar

$$f = 0$$

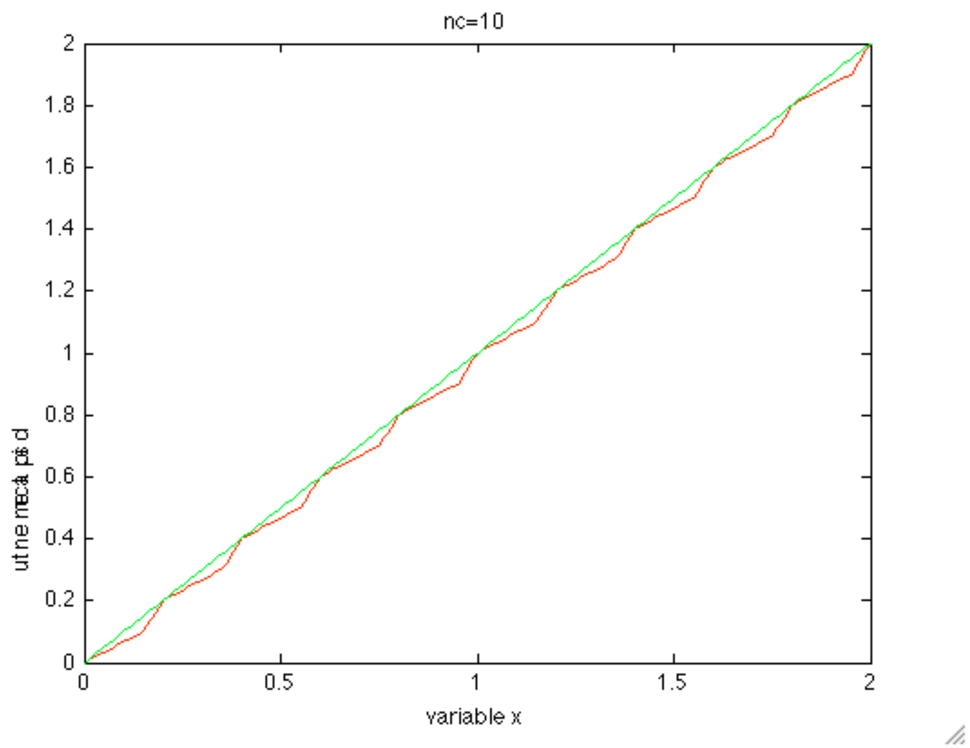


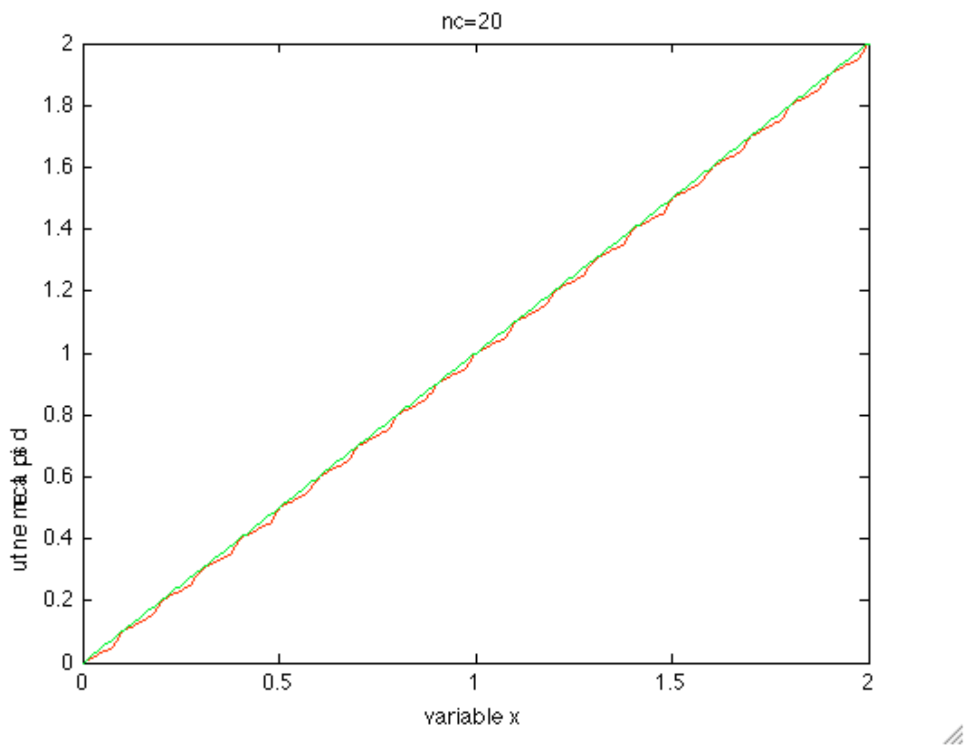


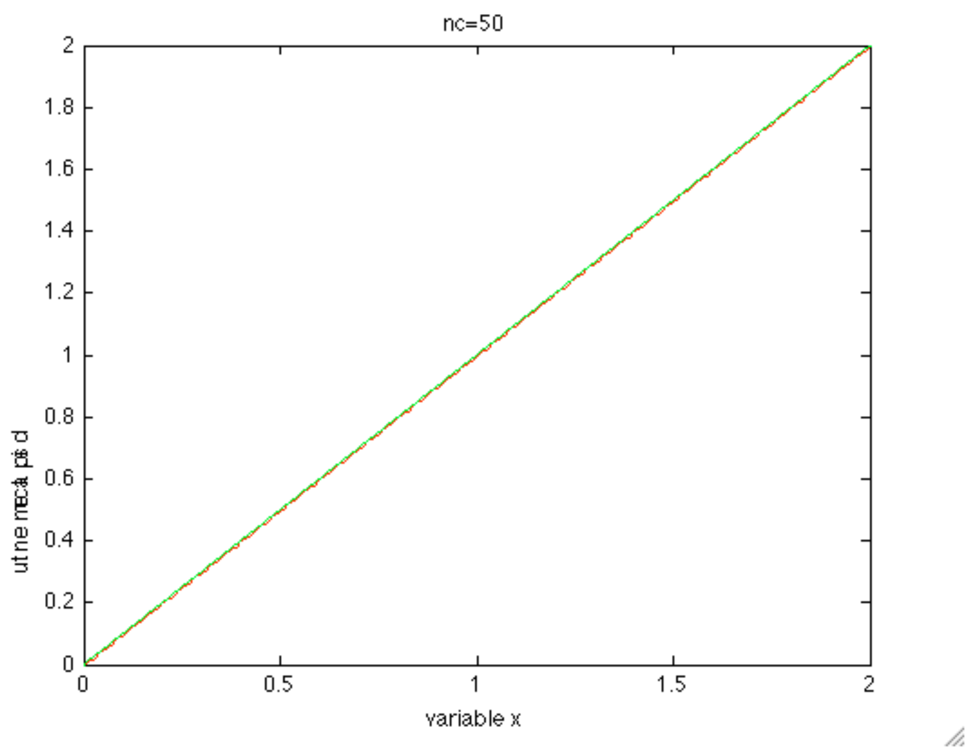


1D periodic bar

Convergence to the homogenized solution

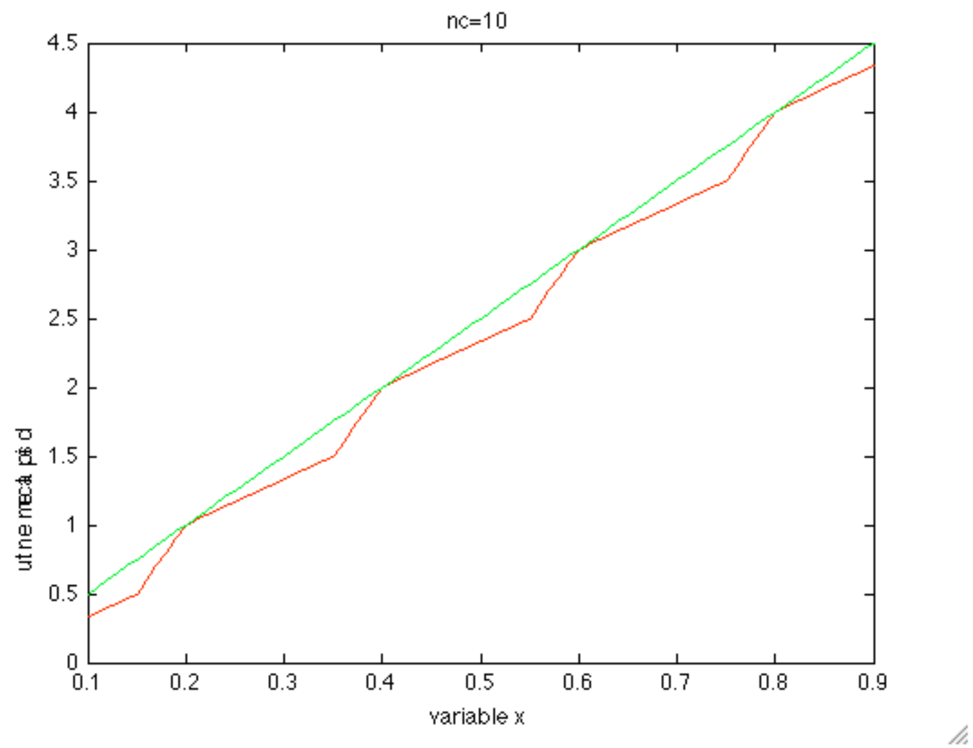


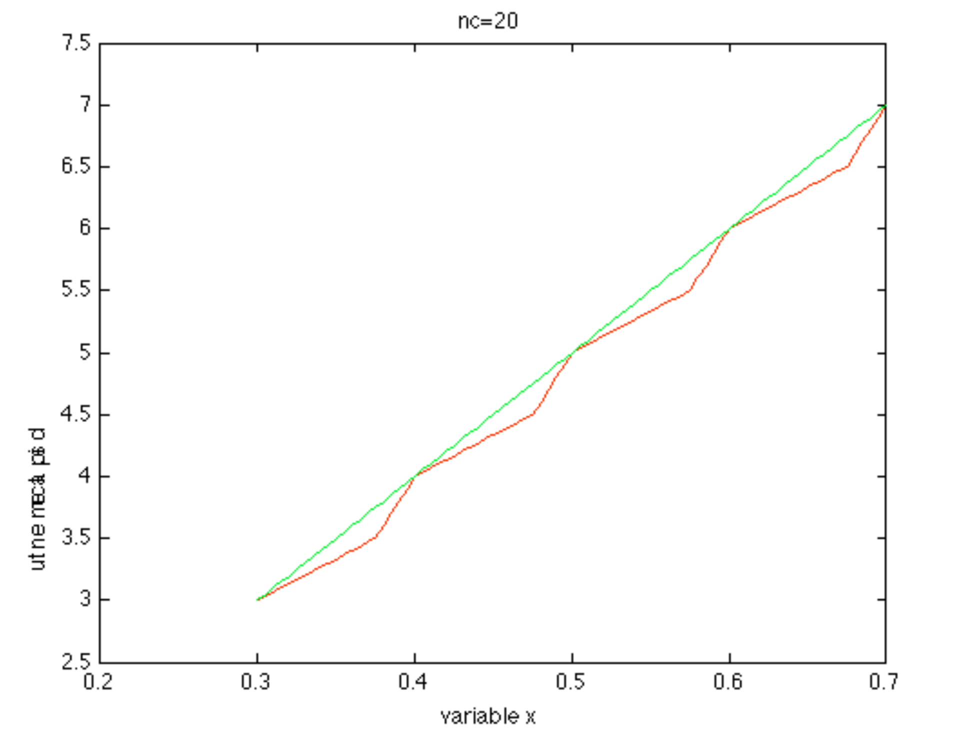


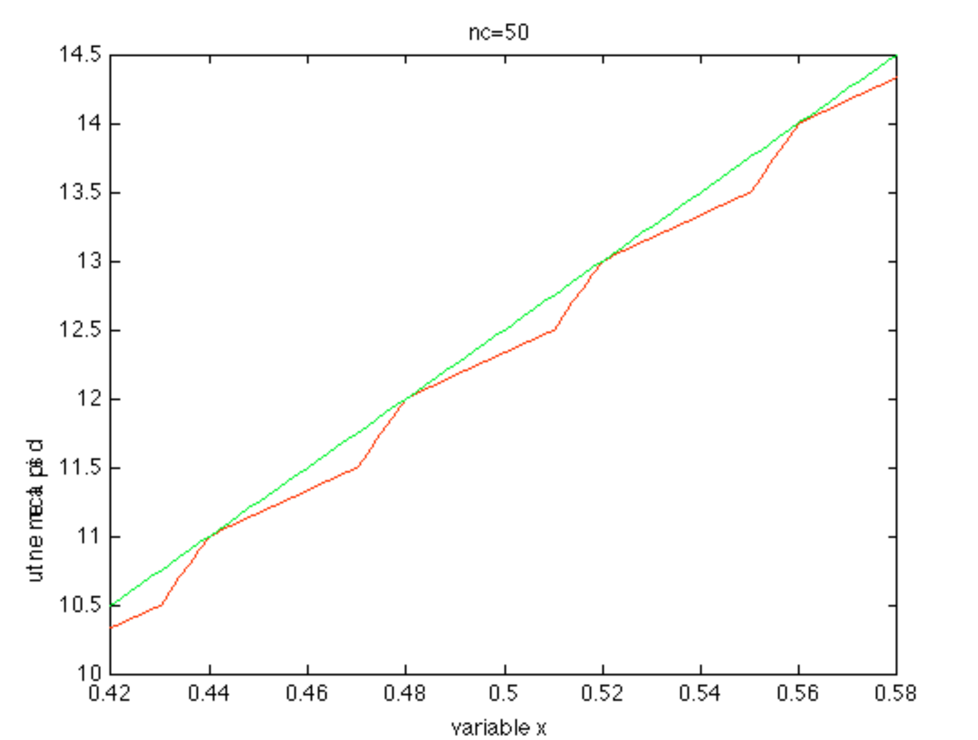


1D periodic bar

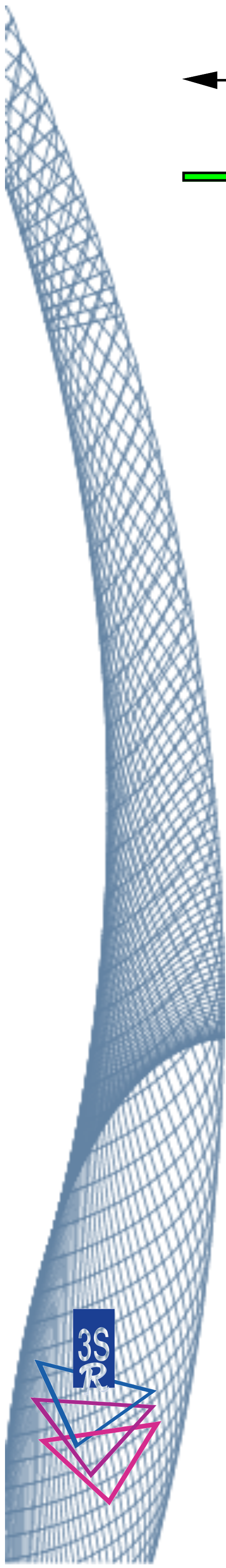
Zoom X Nc around $x = 0.5$

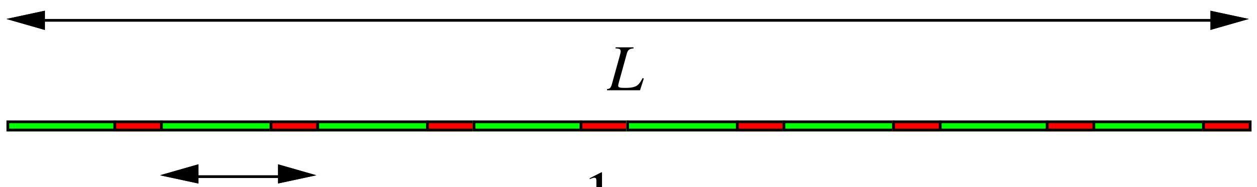




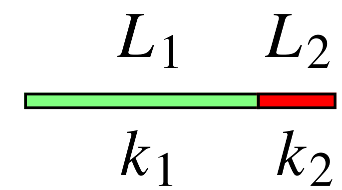


1D periodic bar - Heuristic homogenization





$e = \dfrac{1}{n_c}$, n_c number of cells



At the small scale the normal stress N is almost constant:

$$\frac{dN}{dx} = 0$$

and the variation of the displacement on a length equal to the period eL corresponds to the macroscopic strain:

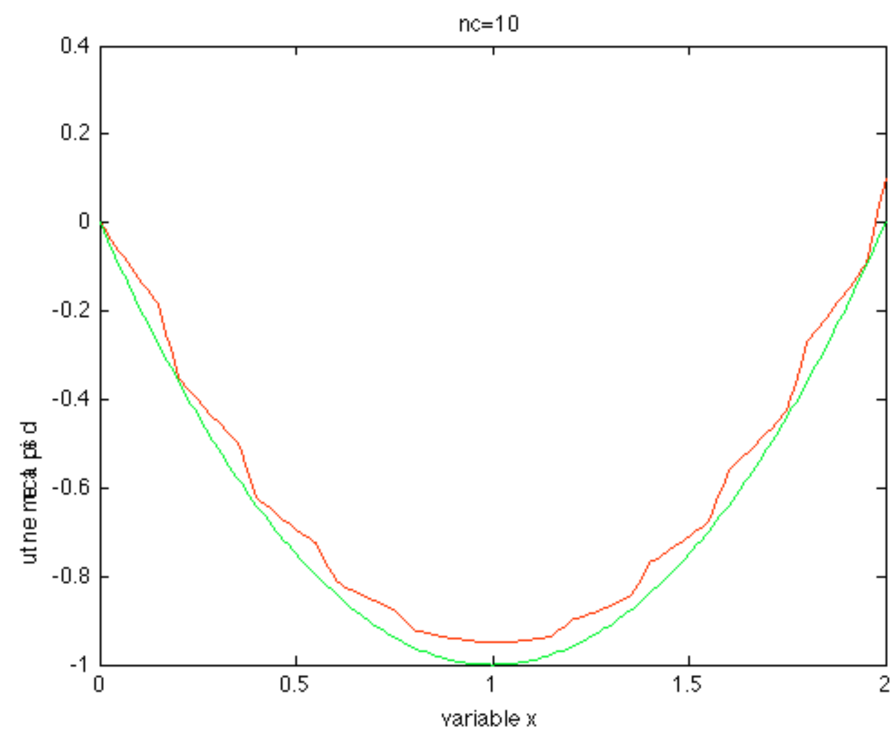
$$u\left(x + eL\right) - u\left(x\right) = \varepsilon^M eL$$

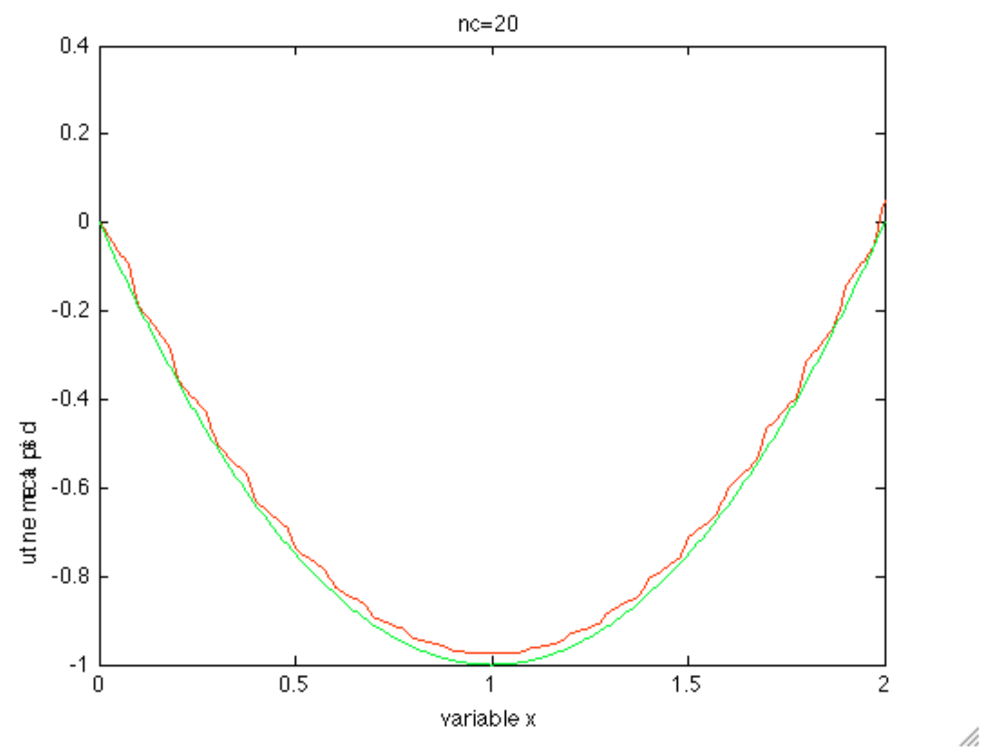
The integration of the constitutive equation $N = k^e\left(x\right)\dfrac{du}{dx}$ yields:

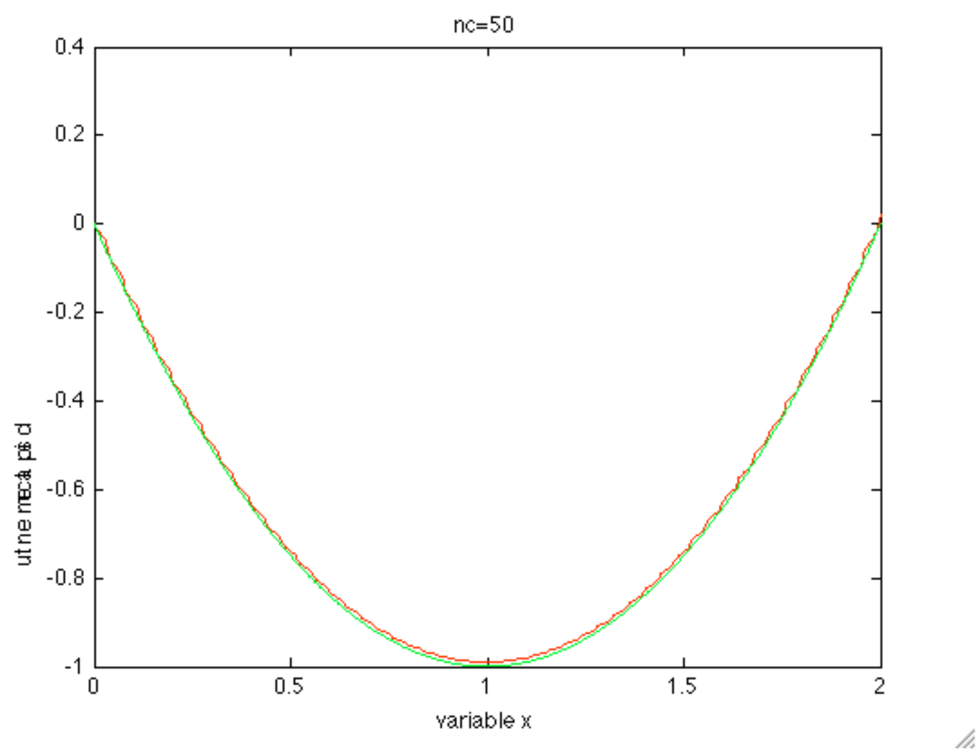
$$\varepsilon^M = \frac{N}{eL}\int_x^{x+eL} \frac{1}{k^e\left(\xi\right)}\,\mathrm{d}\xi = \frac{1}{L}\left(\frac{L_1}{k_1} + \frac{L_2}{k_2}\right) N$$

Which is the **equivalent macroscopic constitutive equation**

1D periodic bar - Case of $f = -2$





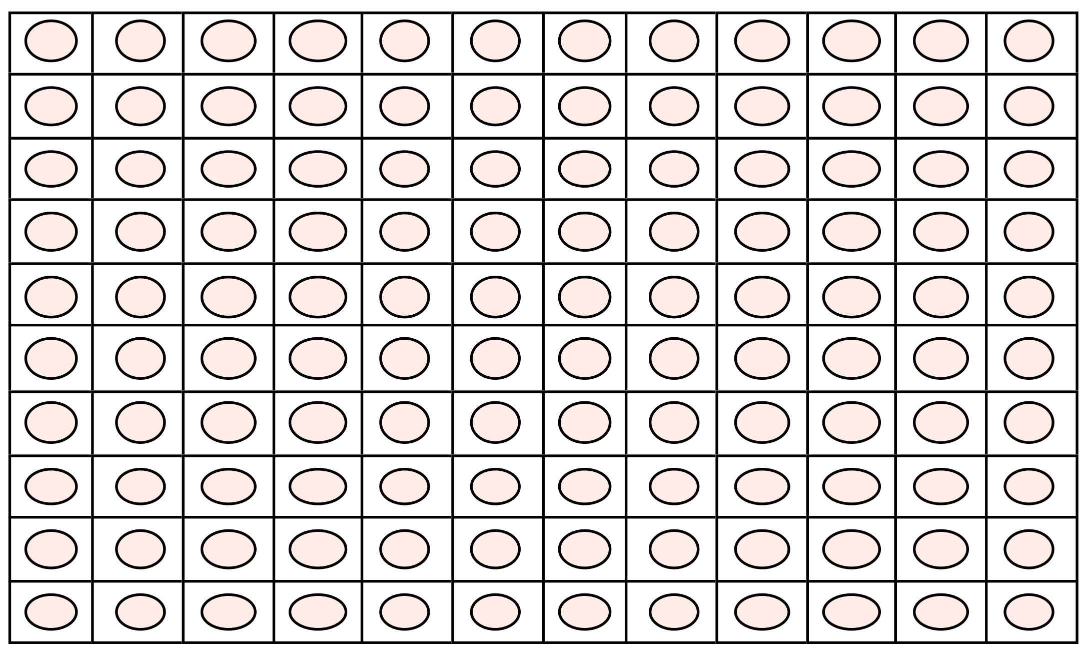


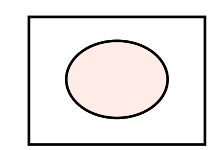
2D Elastic periodic medium

$$\mathrm{div}\sigma^e + \vec{f} = 0 \text{ in } \Omega$$

$$\sigma^e = C\left(\frac{\vec{x}}{e}\right)\epsilon\left(\vec{u}^e\right) \text{ in } \Omega$$

$$+ \text{ boundary conditions on } \partial\Omega$$





Y

The 4th order tensor $C\left(\vec{y}\right)$ is Y periodic

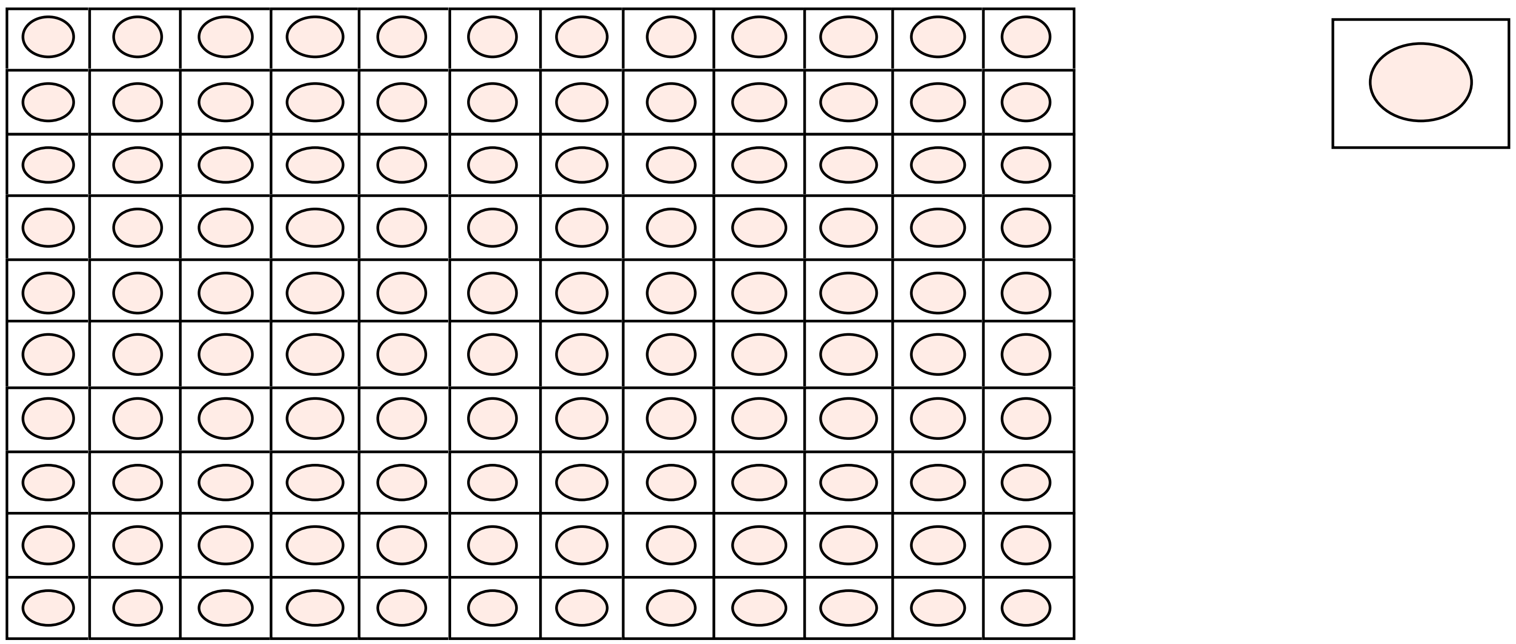
$$e \longrightarrow 0$$

Double scale asymptotic expansion

$$\vec{u}^e(\vec{x}) = \vec{u}^{(0)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e\vec{u}^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2\vec{u}^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots$$

The $\vec{u}^{(k)}(\vec{x},\vec{y})$ are Y periodic so the $\vec{u}^{(k)}\left(\vec{x}, \frac{\vec{x}}{e}\right)$ are almost eY periodic

Slow x and fast y variables



Double scale asymptotic expansion

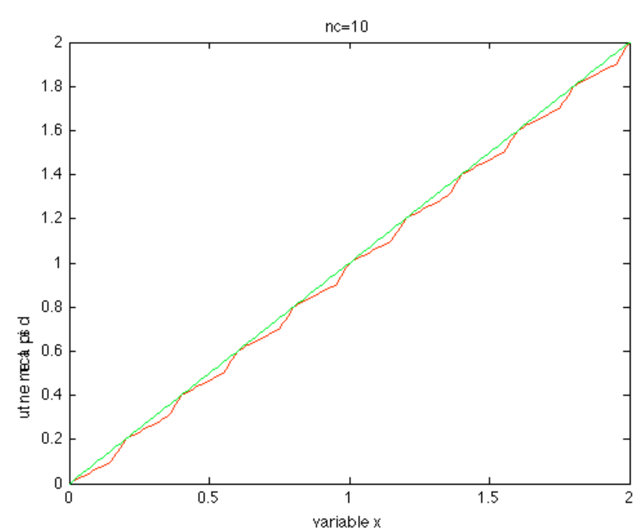
It turns out that $\vec{u}^{(0)}$ does not depend on $\vec{y}$

$$\vec{u}^{e}\left(\vec{x}\right) = \vec{u}^{(0)}\left(\vec{x}\right) + e\vec{u}^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^{2}\vec{u}^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots$$

$\vec{u}^{(0)}\left(\vec{x}\right)$ is the macroscopic displacement field

and, up to higher terms, the displacement $\vec{u}^{e}\left(\vec{x}\right) = \vec{u}^{(0)}\left(\vec{x}\right) + e\vec{u}^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right)$

is equal to the macroscopic one + a small correction presenting fast variations



Double scale asymptotic expansion

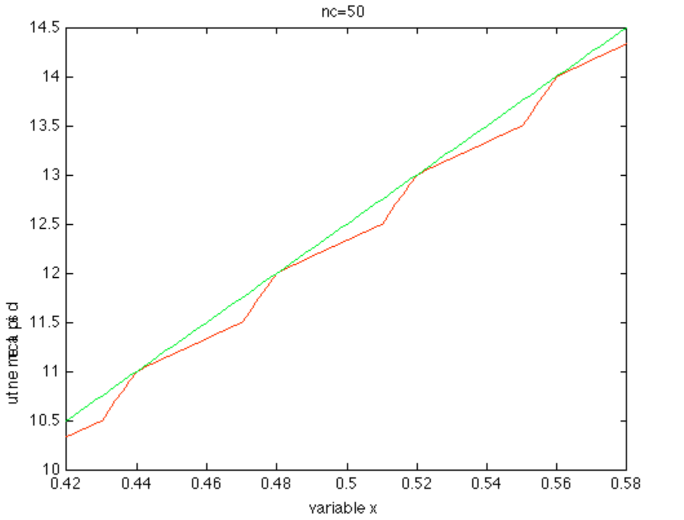
$$\vec{u}^e = \vec{u}^{(0)}\left(\vec{x}\right) + e\vec{u}^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2\vec{u}^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots$$

$$\nabla\vec{u}^e = \nabla^x\vec{u}^{(0)} + \nabla^y\vec{u}^{(1)} + e\left(\nabla^x\vec{u}^{(1)} + \nabla^y\vec{u}^{(2)}\right) + \cdots$$

$$\epsilon\left(\vec{u}^e\right) = \epsilon^x\left(\vec{u}^{(0)}\right) + \epsilon^y\left(\vec{u}^{(1)}\right) + e\left(\epsilon^x\left(\vec{u}^{(1)}\right) + \epsilon^y\left(\vec{u}^{(2)}\right)\right) + \cdots$$

$$\sigma^e = \sigma^{(0)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e\sigma^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2\sigma^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots$$

$\sigma^{(k)}\left(\vec{x}, \vec{y}\right)$ are Y periodic



Expansion of the equilibrium equation

$$\sigma^e = \sigma^{(0)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e\sigma^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2\sigma^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots$$

$$\mathrm{div}\sigma^e = \frac{1}{e}\mathrm{div}^y\sigma^{(0)} + \mathrm{div}^x\sigma^{(0)} + \mathrm{div}^y\sigma^{(1)} + e\left(\cdots\right)$$

The balance equation $\mathrm{div}\sigma^e + \vec{f} = 0$ expands into:

$$\frac{1}{e}\mathrm{div}^y\sigma^{(0)} + \mathrm{div}^x\sigma^{(0)} + \mathrm{div}^y\sigma^{(1)} + e\left(\cdots\right) + \vec{f} = 0$$

which, by identification of the terms of same power, yields:

$$\mathrm{div}^y\sigma^{(0)} = 0$$

$$\mathrm{div}^x\sigma^{(0)} + \mathrm{div}^y\sigma^{(1)} + \vec{f} = 0$$

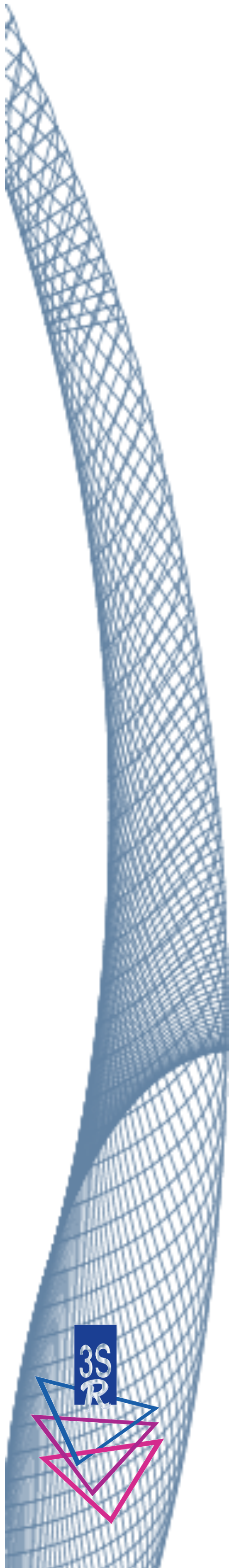
Macroscopic balance equation

$$\mathrm{div}^x\left\langle\sigma^{(0)}\right\rangle + \vec{f} = 0$$

Macroscopic mean stress

$$\left\langle\sigma^{(0)}\right\rangle = \frac{1}{|Y|}\int_Y \sigma^{(0)}\left(\vec{x}, \vec{y}\right)\,\mathrm{d}y$$

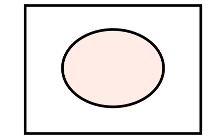
Self equilibrium problem



$$\mathrm{div}^y \sigma^{(0)} = 0 \text{ in } Y$$

$$\sigma^{(0)} = C(\vec{y}) \left(\epsilon^x \left(\vec{u}^{(0)} \right) + \epsilon^y \left(\vec{u}^{(1)} \right) \right) \text{ in } Y$$

$$+ \text{ periodic boundary conditions on } \partial Y$$



The unknowns are $\sigma^{(0)}$ and $\vec{u}^{(1)}$

the datum is the macroscopic strain $\epsilon^x \left(\vec{u}^{(0)} \right)$

The solving of this problem yields $\vec{u}^{(1)}$ and $\sigma^{(0)}$

and by averaging, the macroscopic stress $\left\langle \sigma^{(0)} \right\rangle$

Which defines the equivalent macroscopic constitutive relation:

$$\epsilon^x \left(\vec{u}^{(0)} \right) \longrightarrow \vec{u}^{(1)} \text{ and } \sigma^{(0)} \longrightarrow \left\langle \sigma^{(0)} \right\rangle$$

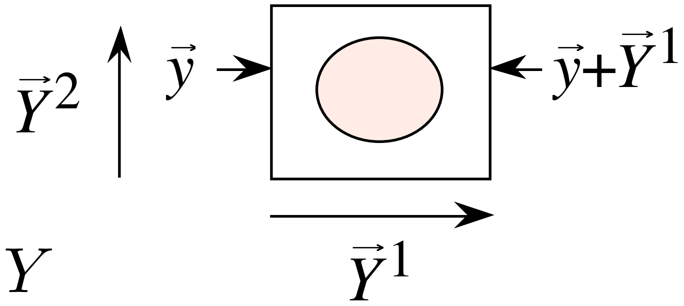
Self equilibrium problem - second form

$$\vec{\hat{u}}^{(1)}\left(\vec{y}\right) = \epsilon^{x}\left(\vec{u}^{(0)}\right).\vec{y} + \vec{u}^{(1)}\left(\vec{x},\vec{y}\right)$$

then:

$$\nabla^{y}\vec{\hat{u}}^{(1)} = \epsilon^{x}\left(\vec{u}^{(0)}\right) + \nabla^{y}\vec{u}^{(1)}$$

the problem for $\vec{\hat{u}}^{(1)}$ reads:



$$\operatorname{div}^{y}\sigma^{(0)} = 0 \text{ in } Y$$

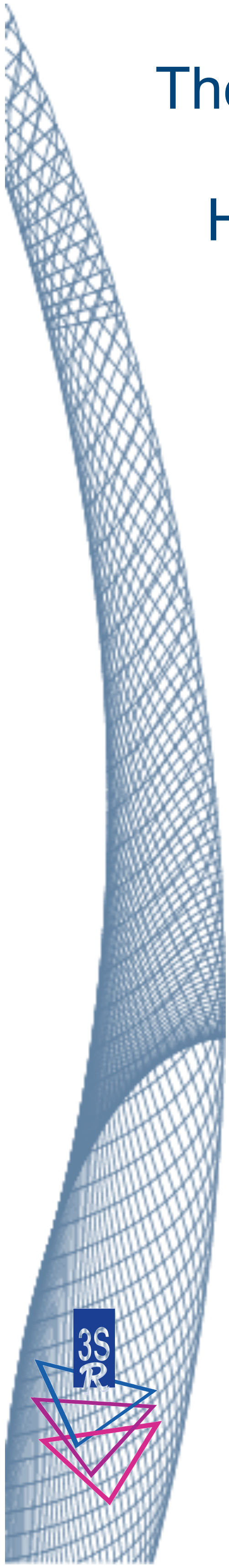
$$\sigma^{(0)} = C\left(\vec{y}\right)\epsilon^{y}\left(\vec{\hat{u}}^{(1)}\right) \text{ in } Y$$

$$\vec{\hat{u}}^{(1)}\left(\vec{y}+\vec{Y}^{i}\right) - \vec{\hat{u}}^{(1)}\left(\vec{y}\right) = \epsilon^{x}\left(\vec{u}^{(0)}\right).\vec{Y}^{i} \text{ on } \Gamma^{i}\,,\ i = 1,2$$

Hill's lemma

$$\left\langle \sigma^{(0)} \right\rangle : \epsilon^{x}\left(\vec{u}^{(0)}\right) = \left\langle \sigma^{(0)} : \epsilon^{y}\left(\vec{\hat{u}}^{(1)}\right) \right\rangle$$

Self equilibrium problem - third form on the real cell



The small parameter $e = \ell / L$ is somehow arbitrary

Homothety from the expanded cell Y to the real cell $Y^e_{\vec{x}}$ located by $\vec{x}$

$$\vec{y} \leftrightarrow \vec{\xi} = e\vec{y}$$

Change of unknowns

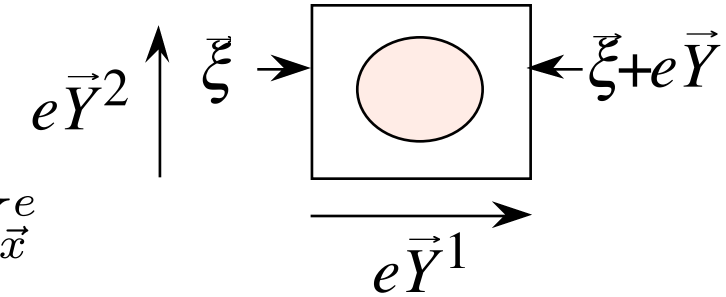
$$\tilde{\vec{u}}^{(1)}\left(\vec{\xi}\right) = e\hat{\vec{u}}^{(1)}\left(\frac{\vec{\xi}}{e}\right) = e\left(\epsilon^x\left(\vec{u}^{(0)}\right).\frac{\vec{\xi}}{e} + \vec{u}^{(1)}\left(\vec{x}, \frac{\vec{\xi}}{e}\right)\right)$$

$$\tilde{\sigma}^{(0)}\left(\vec{\xi}\right) = \sigma^{(0)}\left(\vec{x}, \frac{\vec{\xi}}{e}\right)$$

$\tilde{\vec{u}}^{(1)}$ and $\tilde{\sigma}^{(0)}$ are solutions of the problem set on the real cell $C_{\vec{x}}$

$$\mathrm{div}^{\xi}\tilde{\sigma}^{(0)} = 0 \text{ in } Y^e_{\vec{x}}$$

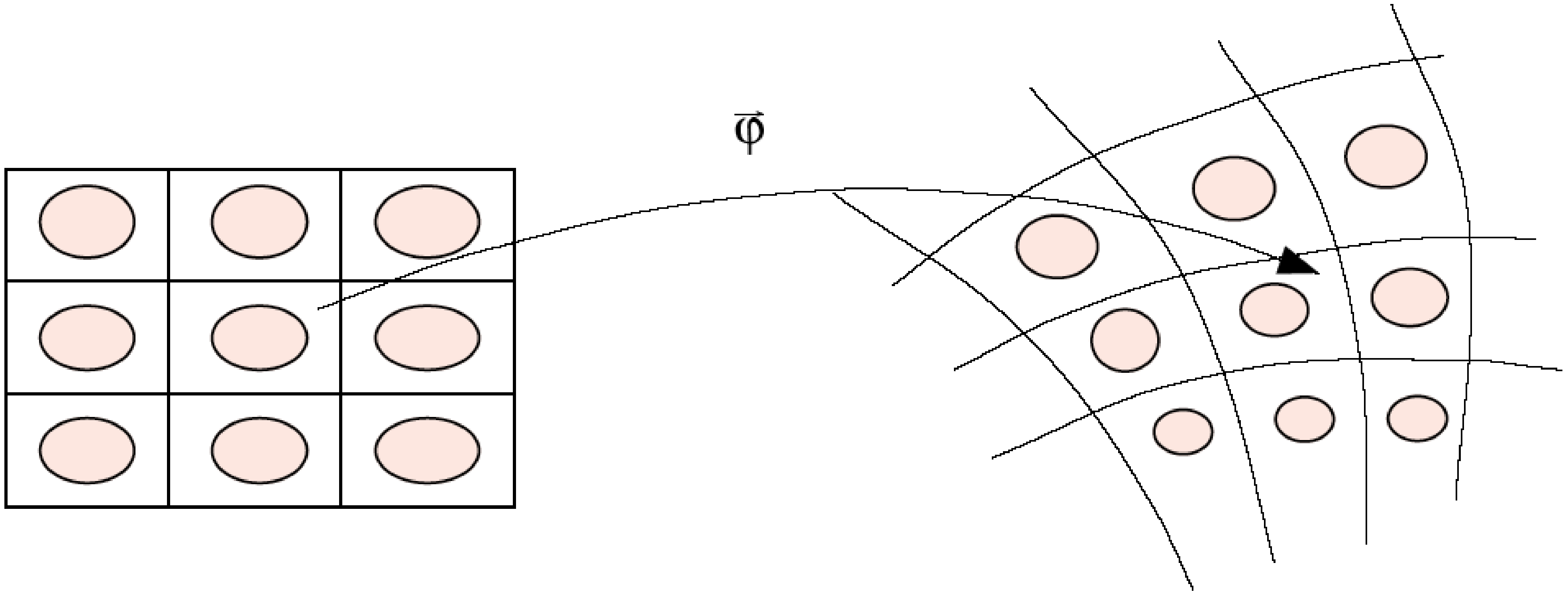
$$\tilde{\sigma}^{(0)} = C\left(\frac{\vec{\xi}}{e}\right)\epsilon^{\xi}\left(\tilde{\vec{u}}^{(1)}\right) \text{ in } Y^e_{\vec{x}}$$



$$\tilde{\vec{u}}^{(1)}\left(\vec{\xi} + e\vec{Y}^i\right) - \tilde{\vec{u}}^{(1)}\left(\vec{\xi}\right) = \epsilon^x\left(\vec{u}^{(0)}\right).e\vec{Y}^i \text{ on } \gamma^i_{\vec{x}}\,,\ i = 1, 2$$

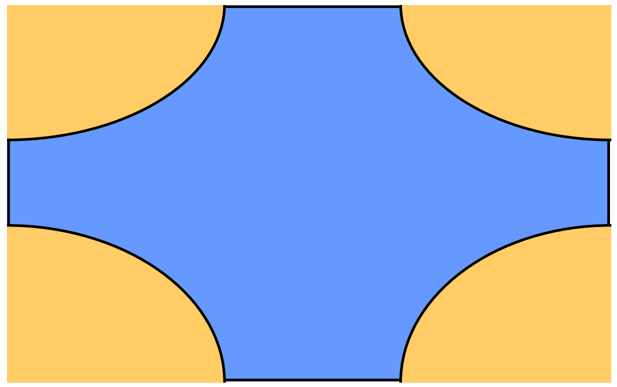
Remak: heuristic method for the bar

Quasi periodic media



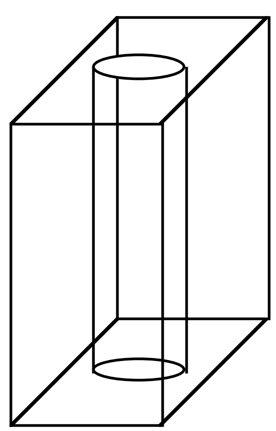
Applications
Geometrically quasiperiodic media
Elastoplastic periodic media
Large strain framework

Multiphysics and multi parameter modellings



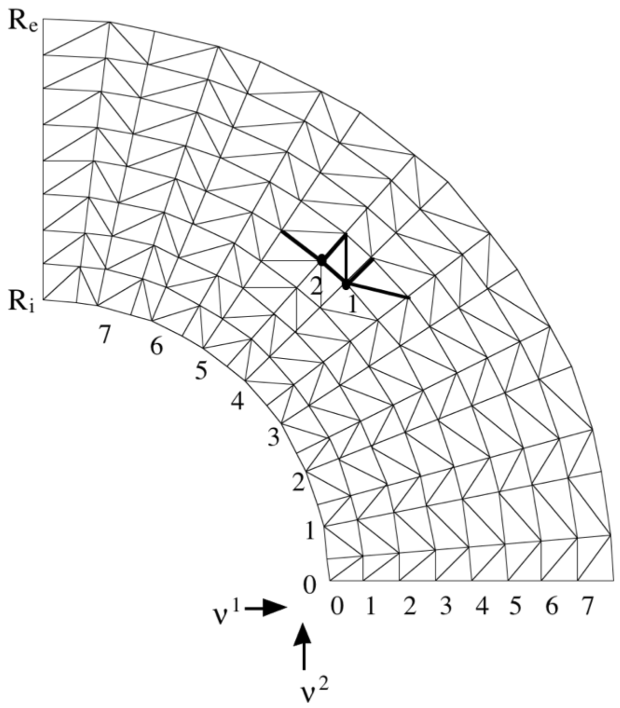
Darcy's law
Biot's modelling of the consolidation
Large strains
Viscosity of the fluid $\simeq e^2$

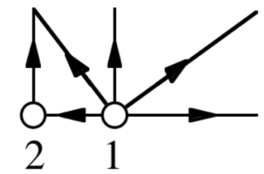
Periodic plates or beams



Medium reinforced by slender fibers

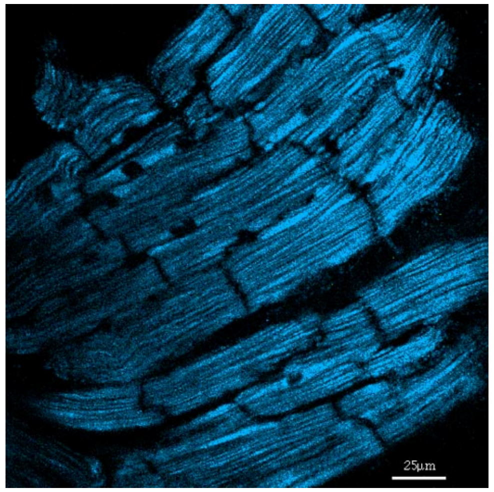
Continuous modelling of discrete stuctures



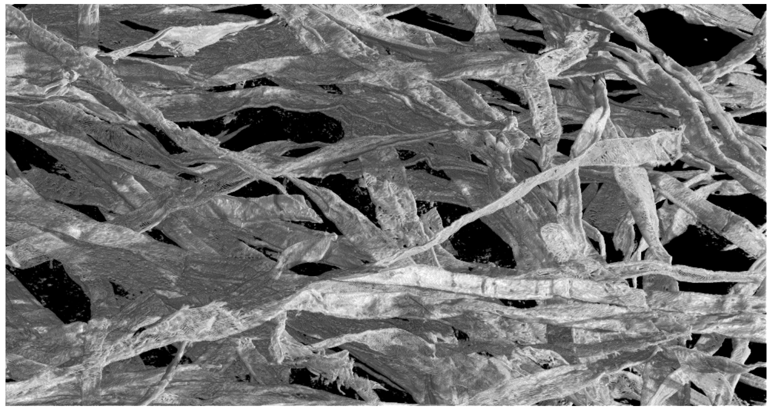


$$\tilde{n} = \left(\nu^1, \nu^2, n\right)$$

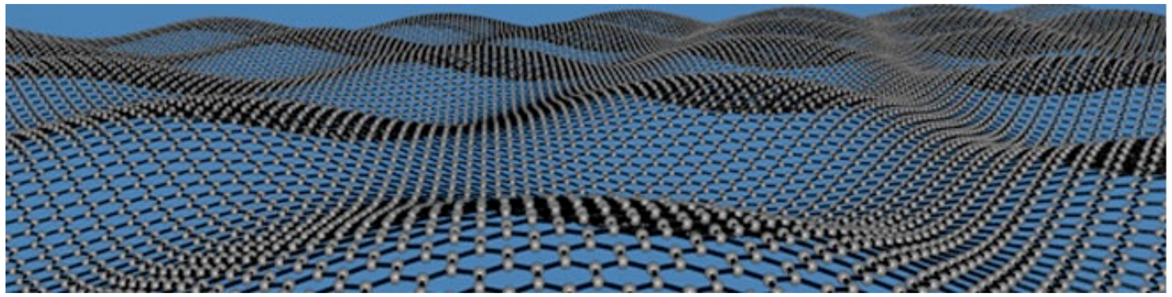
$$\vec{u}^{\tilde{n}} = \vec{u}^{(0)}\left(e\nu^1, e\nu^2\right) + e\vec{u}^{(1)n}\left(e\nu^1, e\nu^2\right) + \cdots$$



Muscle



Paper



Graphene

Intermediate conclusions and remarks

The homogenization method of periodic media is robust in the sense that, starting from the description of the macroscopic scale, it enables to define the equivalent fields at the macroscopic scale and the equations that govern them and that with the only hypothesis of the double scale expansion.

The approximation of the real displacement field is the macroscopic field + a small corrector presenting variations at the micro scale

The macroscopic constitutive equation is given by the solving of a self equilibrium problem on the elementary cell

The macroscopic constitutive equation is invariant under a change of scale in the self equilibrium problem

Hill's lemma is satisfied

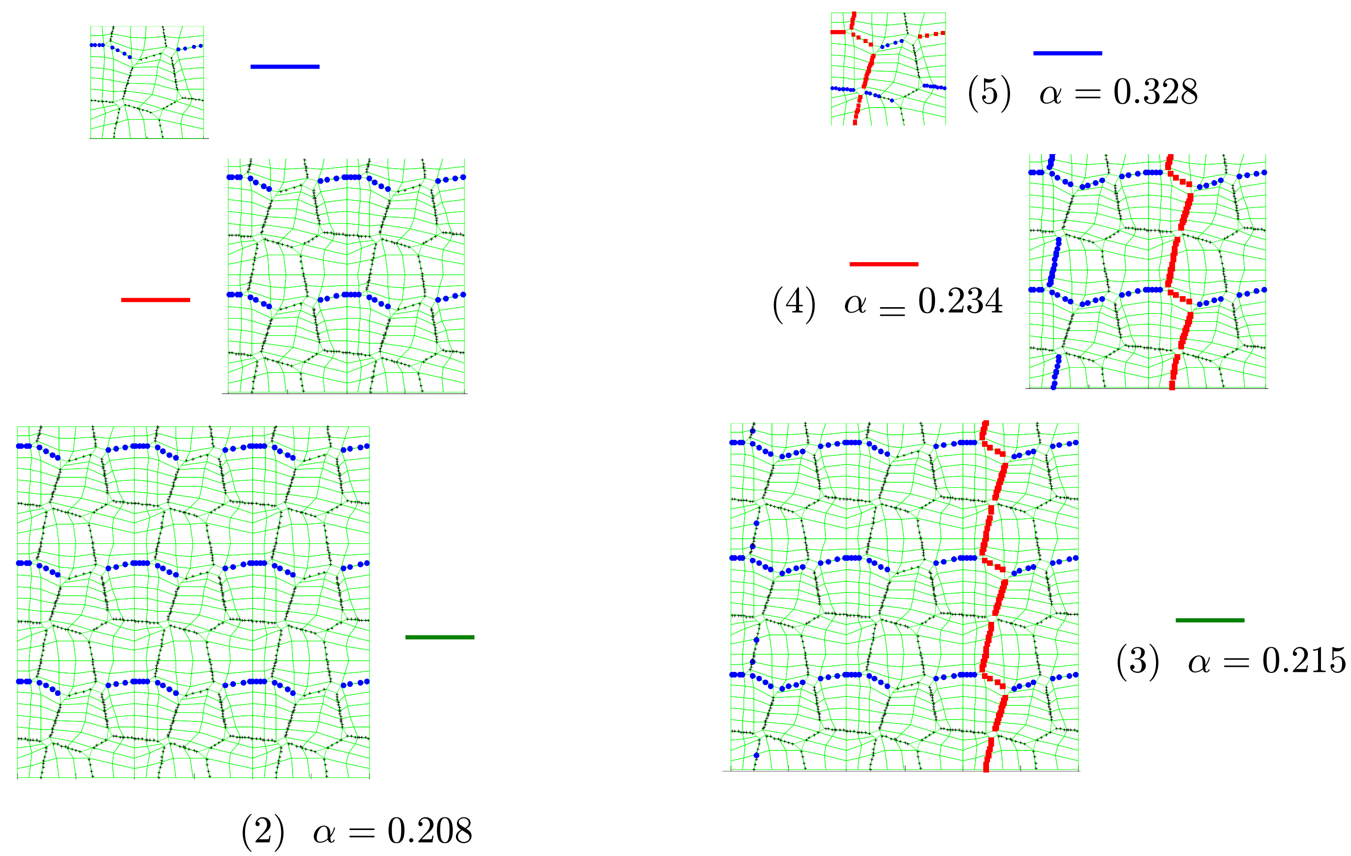
Limits of the methods

Loss of scale separation

wave propagation

assumed macroscopic quantities that varies at the microscale

Bifurcation



From G. Bilbie's PhD Thesis

Parallel with experiments - Representative volume element

Triaxial, biaxial, œdometric tests are designed to be “homogeneous” test

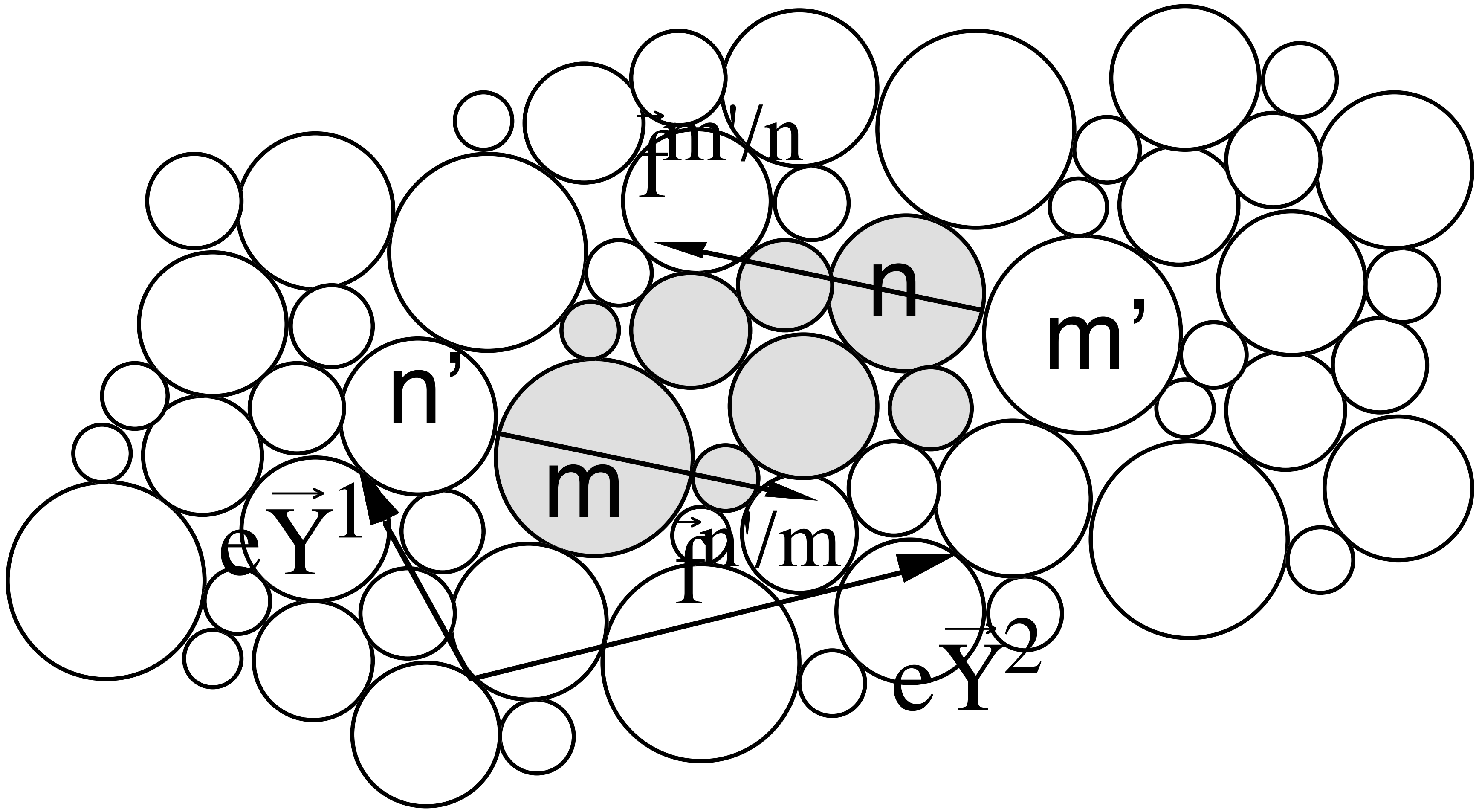
The solving of the self equilibrium problem on the elementary cell

is a numerical experiment analogous to the physical one

Hill's definiton of the RVE (JMPS 1963)

This phrase will be used when referring to a sample that (a) is structurally entirely typical of the whole mixture on average, and (b) contains a sufficient number of inclusions for the apparent overall moduli to be effectively independent of the surface values of traction and displacement, so long as these values are “macroscopically uniform”. That is, they fluctuate about a mean with a wavelength small compared with the dimensions of the sample, and the effects of such fluctuations become insignificant within a few wavelengths of the surface. The contribution of this surface layer to any average can be made negligible by taking the sample large enough.

Granular materials



$$\vec{x}^{m'} = \vec{x}^{m} + eF.\vec{Y}^{1}$$

Thank you for your attention