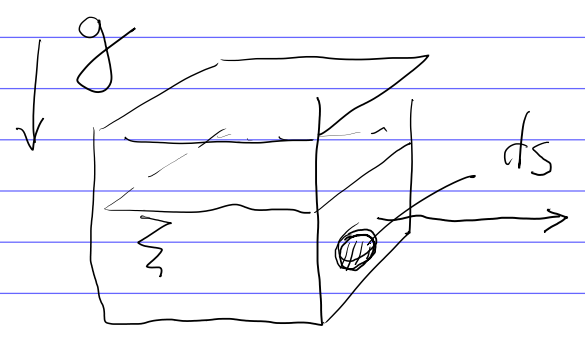
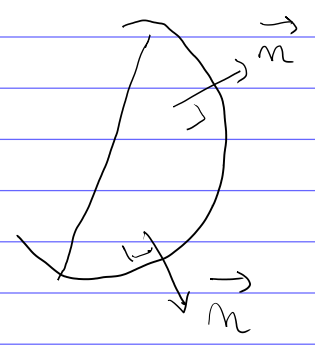


Notion de pression



$$\vec{F} (\text{fluide} \rightarrow \text{paroi}) = \int_S p(x, y, z) \vec{dS}(x, y, z)$$

$(d\vec{S} = ds \vec{n})$
 \uparrow
 normale unitaire



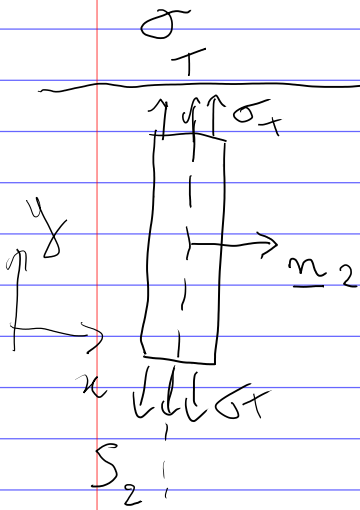
$$\vec{F} = \int_S p(\underline{x}) n(\underline{x}) ds$$

densité de force par unité de surface $\perp ds$



σ_T contrainte de traction

$$\vec{T} = \sigma_T \vec{n}_1$$



$$\vec{T} = 0 \vec{e}_1 = 0 \vec{n}_2$$

Théorème de Cauchy.

T : vecteur contrainte

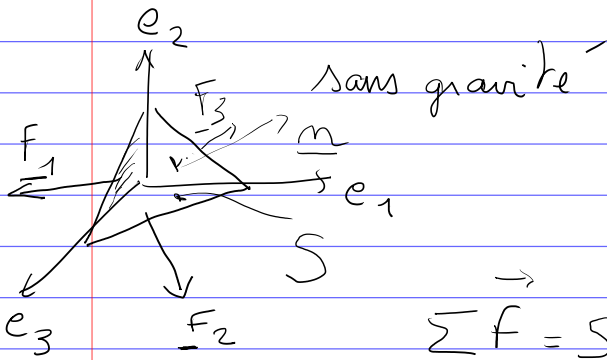
$$\underline{T}(\underline{x}, \underline{n}) = \underline{\underline{\sigma}} \underline{n}$$

↑ ↑
position direction

σ tenseur de contrainte

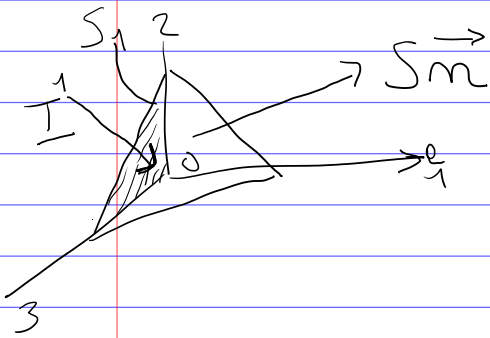
↑
matrice

Démo :



$$\vec{T}^1(x=0, \underline{n}=-\underline{e}_1)$$

$$\sum \underline{F} = S_1 \underline{T}^1 + S_2 \underline{T}^2 + S_3 \underline{T}^3$$



$$+ S \underline{T} = \underline{0} \quad (1)$$

↑
équilibre

On a

$$S_1 = \underbrace{S \underline{n}}_{\underline{s}} \cdot \underline{e}_1$$

$$S \underline{T} + S \underline{T}^1 \underline{n} \cdot \underline{e}_1 + \dots = \underline{0}$$

$$\underline{T}(\underline{n}) = \begin{pmatrix} T_1^1 & T_1^2 & T_1^3 \\ T_2^1 & T_2^2 & T_2^3 \\ T_3^1 & T_3^2 & T_3^3 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

direction
qcq

vector constraint
dans la direction 1

Preuve de l'existence
du tenseur de contrainte $\underline{\underline{\sigma}}$

Sur la face (dx_1, dx_2)

$$\vec{n} = \vec{e}_3$$

$$\vec{T} = \underline{\underline{\sigma}} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Gradient ∇p

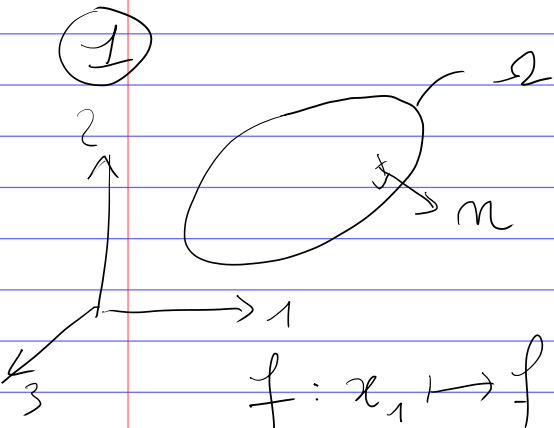
Divergent - $\text{div}(\vec{v})$

Ici on va avoir

tenseur $\leftarrow \nabla \vec{v}$ (gradient d'un vecteur)

vecteur $\leftarrow \nabla \cdot \underline{\underline{\sigma}}$ (divergence d'un tenseur)

Th. de la divergence.

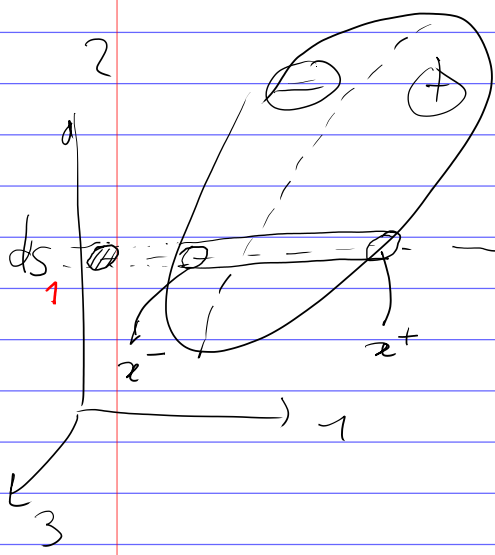


$$f: x_1 \mapsto f(x_1)$$

Proposition:

$$\int_{\Omega} \frac{df}{dx_1} dV = \int_{\partial\Omega} f n_1 dS$$

\uparrow
contour



$$\frac{df}{dx} = \left(\frac{df}{dx} \right)$$

$$= \iint_{\mathcal{R}} [f(x^+) - f(x^-)] ds$$

on a $ds = \cancel{ds^+} \cdot \frac{e_1}{1} = \cancel{ds^-} \cdot \frac{e_1}{1}$

$$= \iint_{\mathcal{R}} f(x^+) ds^+ \frac{e_1}{1} - f(x^-) ds^- \frac{e_1}{1} ds$$

$$= \iint_{\mathcal{R}} f(x) \cancel{ds} \frac{ds_1}{1} = \int_{\partial\Omega} f(x) \cancel{ds} \frac{ds_1}{1}$$

J'ai montré

$$\int_{\Omega} \frac{\partial f}{\partial x_1} d\sigma = \iint_{\partial\Omega} f n_1 ds$$

Corollaire 1 :

$$\text{Si } f = f(x, y, z)$$

$$\text{alors } \int_{\Omega} \left(\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3} \right) d\sigma = \int_{\partial\Omega} (f n_x + f n_y + f n_z) ds$$

Corollaire 2 :

Si 3 fonctions f_x, f_y, f_z de (x, y, z)

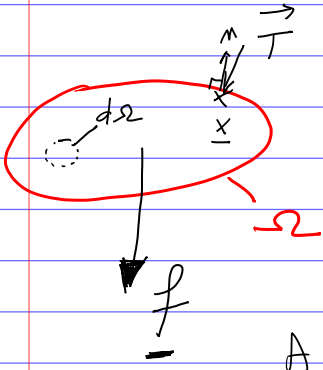
$$\int_{\Omega} \left(\frac{\partial f_x}{\partial x_1} + \frac{\partial f_y}{\partial x_2} + \frac{\partial f_z}{\partial x_3} \right) d\sigma = \int_{\partial\Omega} (f_x n_x + f_y n_y + f_z n_z) ds$$

autrement dit si $\underline{f} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$

$$\int_{\Omega} \nabla \cdot \underline{f} d\sigma = \int_{\partial\Omega} \underline{f} \cdot \underline{n} ds$$

\uparrow
div(f)

① Equation d'équilibre



\underline{f}_v : force de volume
(ex. gravité)

\underline{T} : force de surface

A l'équ. $\Sigma \vec{F} = \vec{0}$

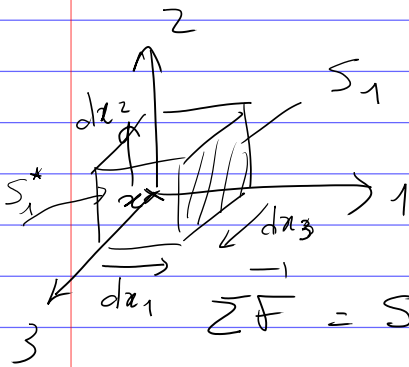
$$\Rightarrow \int_{\Omega} \underline{f}_v d\tau + \int_{\partial\Omega} \underline{T} ds = \underline{0}$$

Prendons $\Omega \rightarrow 0$

$$\int_{\Omega} \underline{f}_v d\tau + \int_{\partial\Omega} \underline{\sigma} \underline{n} ds = \underline{0}$$

$$\int_{\Omega} (\underline{f}_v + \nabla \cdot \underline{\sigma}) d\tau = 0 \quad (\forall \Omega)$$

$$\Rightarrow \boxed{\underline{f}_v + \nabla \cdot \underline{\sigma} = \underline{0}}$$



vecteur contrainte
dans direction
 \underline{e}_1

parce que

$$\underline{\sigma}(\underline{e}_1) = -\underline{\sigma}^T \underline{e}_1$$

$$\begin{aligned} \underline{\Sigma F} = & S_1 (\underline{T}^1(\underline{x} + d\underline{x}_1) - \underline{T}^1(\underline{x})) \\ & + S_2 (\underline{T}^2(\underline{x} + d\underline{x}_2) - \underline{T}^2(\underline{x})) \\ & + S_3 (\underline{T}^3(\underline{x} + d\underline{x}_3) - \underline{T}^3(\underline{x})) \\ & + \underbrace{\|d\underline{x}_1\| \|d\underline{x}_2\| \times \|d\underline{x}_3\|}_{\text{volume}} \times \underline{f}_v = 0 \end{aligned}$$

$$\text{or } \underline{\underline{\sigma}} = \begin{bmatrix} \underline{T}^1 & \underline{T}^2 & \underline{T}^3 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$\begin{aligned} & \underline{T}^3(\underline{x} + d\underline{x}_3) \\ & - \underline{T}^3(\underline{x}) \end{aligned}$$

$$\Rightarrow \underline{\Sigma F} = \frac{d\underline{T}^1}{\|d\underline{x}_1\|} + \frac{d\underline{T}^2}{\|d\underline{x}_2\|} + \frac{d\underline{T}^3}{\|d\underline{x}_3\|} + \underline{f}_v = 0$$

Rappel: pour un vecteur $\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$,

$$\nabla \cdot \underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

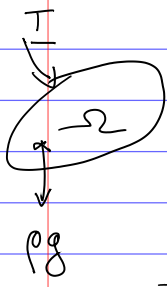
Ici en considérant les colonnes comme les composantes d'un vecteur

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} = \frac{\partial \underline{T}^1}{\partial x_1} + \frac{\partial \underline{T}^2}{\partial x_2} + \frac{\partial \underline{T}^3}{\partial x_3}$$

$$\boxed{\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{f}_v = 0}$$

* Symétrie de $\underline{\underline{\sigma}}$

Moment dû à des efforts surfaciques



The diagram shows a volume Ω with a downward force T at the top and a body force pg acting throughout. The volume is circled.

$$\int_{\Omega} \underline{\underline{\sigma}} \wedge \underline{\underline{pg}} \, dv + \int_{\partial\Omega} \underline{\underline{\sigma}} \wedge \underline{\underline{T}} \, ds = 0$$

↑
à l'équi.

$$= \int_{\Omega} (-) \, dv + \int_{\partial\Omega} \underbrace{\underline{\underline{\sigma}} \wedge \underline{\underline{n}}}_{\underline{\underline{G}}} \, ds = 0$$

divergence de $\underline{\underline{G}}$?!

$$\underline{\underline{x}} \wedge \underline{\underline{\sigma}} = \left[\begin{array}{c} \underline{\underline{x}} \wedge \underline{\underline{T}}^1 \\ \underline{\underline{x}} \wedge \underline{\underline{T}}^2 \\ \underline{\underline{x}} \wedge \underline{\underline{T}}^3 \end{array} \right]$$

$$\rightarrow \frac{\partial (\underline{\underline{x}} \wedge \underline{\underline{T}}^1)}{\partial x_1} = \frac{\partial \underline{\underline{x}}}{\partial x_1} \wedge \underline{\underline{T}}^1 + \underline{\underline{x}} \wedge \frac{\partial \underline{\underline{T}}^1}{\partial x_1}$$

$$\underline{\underline{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{\underline{0}} = \int_{\Omega} \left[\begin{pmatrix} T_3^2 - T_2^3 \\ T_1^3 - T_3^1 \\ T_2^1 - T_1^2 \end{pmatrix} + \underline{\underline{x}} \wedge \left(\frac{\partial \underline{\underline{T}}^1}{\partial x_1} + \frac{\partial \underline{\underline{T}}^2}{\partial x_2} + \frac{\partial \underline{\underline{T}}^3}{\partial x_3} + \underline{\underline{pg}} \right) \right] dv$$

$\frac{\partial \underline{\underline{x}}}{\partial x_1}$

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{\underline{pg}} = 0$$

Le 2nd terme est nul, donc le premier aussi.

$$\begin{bmatrix} T_1^1 & T_1^2 & T_1^3 \\ T_2^1 & T_2^2 & T_2^3 \\ T_3^1 & T_3^2 & T_3^3 \end{bmatrix}$$

Rmq:

$$\underline{\underline{u}} \wedge \underline{\underline{v}} = \begin{bmatrix} 0 & u_3^0 - u_2^0 \\ -u_3^0 & 0 & u_2^0 \\ u_3^0 & -u_2^0 & 0 \end{bmatrix} \underline{\underline{v}}$$

$$\int_{\partial\Omega} (\underline{x} \wedge \underline{\sigma}) \cdot \underline{n} \, ds = \int_{\Omega} [\text{anti-sym}(\underline{\sigma}) + \underline{x} \wedge \underline{\nabla} \cdot \underline{\sigma}] \, dV$$

Résumé :

* \underline{T} : force par unité de surface

* $\underline{T} = \underline{T}(\underline{x}, \underline{n})$

* Il existe $\underline{\sigma}$ tq. $\underline{T} = \underline{\sigma}(\underline{x}) \cdot \underline{n}$

* $\underline{\sigma}$ est symétrique

* Equilibre : $\underline{\nabla} \cdot \underline{\sigma} + \rho \underline{g} = \underline{0}$

Illustrations, cf. p. 20 et suivantes du pdf cours 2.

↳ Conséquence de la symétrie :

$\underline{\underline{\sigma}}$ diagonale dans un certain repère (base propre / diagonalisation)

Remarque : décompositions des matrices

$$\underline{\underline{M}} = \frac{\underline{\underline{M}} + \underline{\underline{M}}^t}{2} + \frac{\underline{\underline{M}} - \underline{\underline{M}}^t}{2}$$

\uparrow symétrique \uparrow antisymétrique

$$\underline{\underline{\sigma}} = \left(\underline{\underline{\sigma}} - \frac{\text{tr}(\underline{\underline{\sigma}})}{3} \underline{\underline{Id}} \right) + \frac{\text{tr}(\underline{\underline{\sigma}})}{3} \underline{\underline{Id}}$$

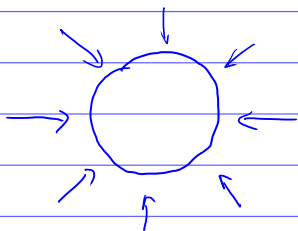
$$\text{tr} \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \sigma_{\text{I}} + \sigma_{\text{II}} + \sigma_{\text{III}}$$

$\uparrow \quad \uparrow \quad \uparrow$
 valeurs propres

$$\begin{pmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{pmatrix}$$

contrainte moyenne.

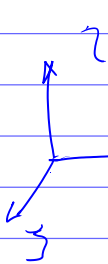
ou : $\underline{\underline{\sigma}} = \text{dev}(\underline{\underline{\sigma}}) + \sigma_m \underline{\underline{Id}}$



Ex. dans un fluide

$$\underline{\underline{\sigma}} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}, \text{dev}(\underline{\underline{\sigma}}) = 0$$

Ex-2 cisaillement



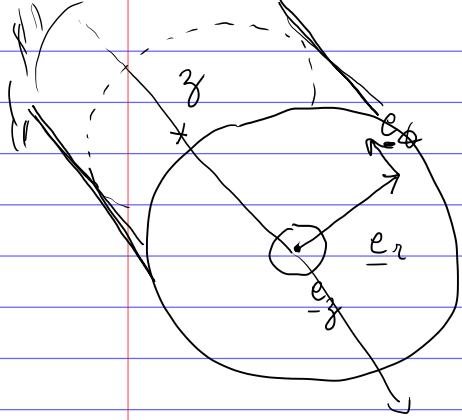
$$\underline{\underline{\sigma}}$$

$$\vec{T}(\vec{e}_2)$$

$$\begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{div}(\underline{\underline{\sigma}}) = \underline{\underline{\sigma}}, \text{tr} \underline{\underline{\sigma}} = 0$$

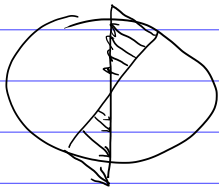
On dit que la contrainte associée à la partie sphérique ($\text{tr} \underline{\underline{\sigma}} = \frac{\text{tr} \underline{\underline{\sigma}}}{3}$) est "isotrope".



1. Examiner les symétries et invariances pour éliminer les termes.

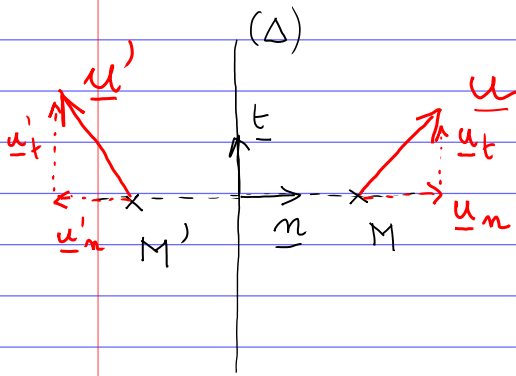
Conditions aux limites :

définir \vec{T} en tout point du contour $\partial\Omega$



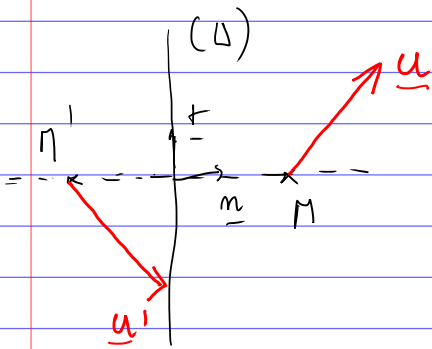
$$\vec{T} = r\alpha \vec{e}_\theta \quad (\text{rappel de RDM})$$

* Annexe. Projections / symmétries



Champs vectoriel
symétrique :

$$\begin{cases} \underline{u}'_n = -\underline{u}_n \\ \underline{u}'_t = \underline{u}_t \end{cases}$$



Champs anti-sym. :

$$\begin{cases} \underline{u}'_n = \underline{u}_n \\ \underline{u}'_t = -\underline{u}_t \end{cases}$$

$$\underline{u}_n = (\underline{u} \cdot \underline{n}) \underline{n}$$

$$\underline{u}_t = \underline{u} - (\underline{u} \cdot \underline{n}) \underline{n}$$