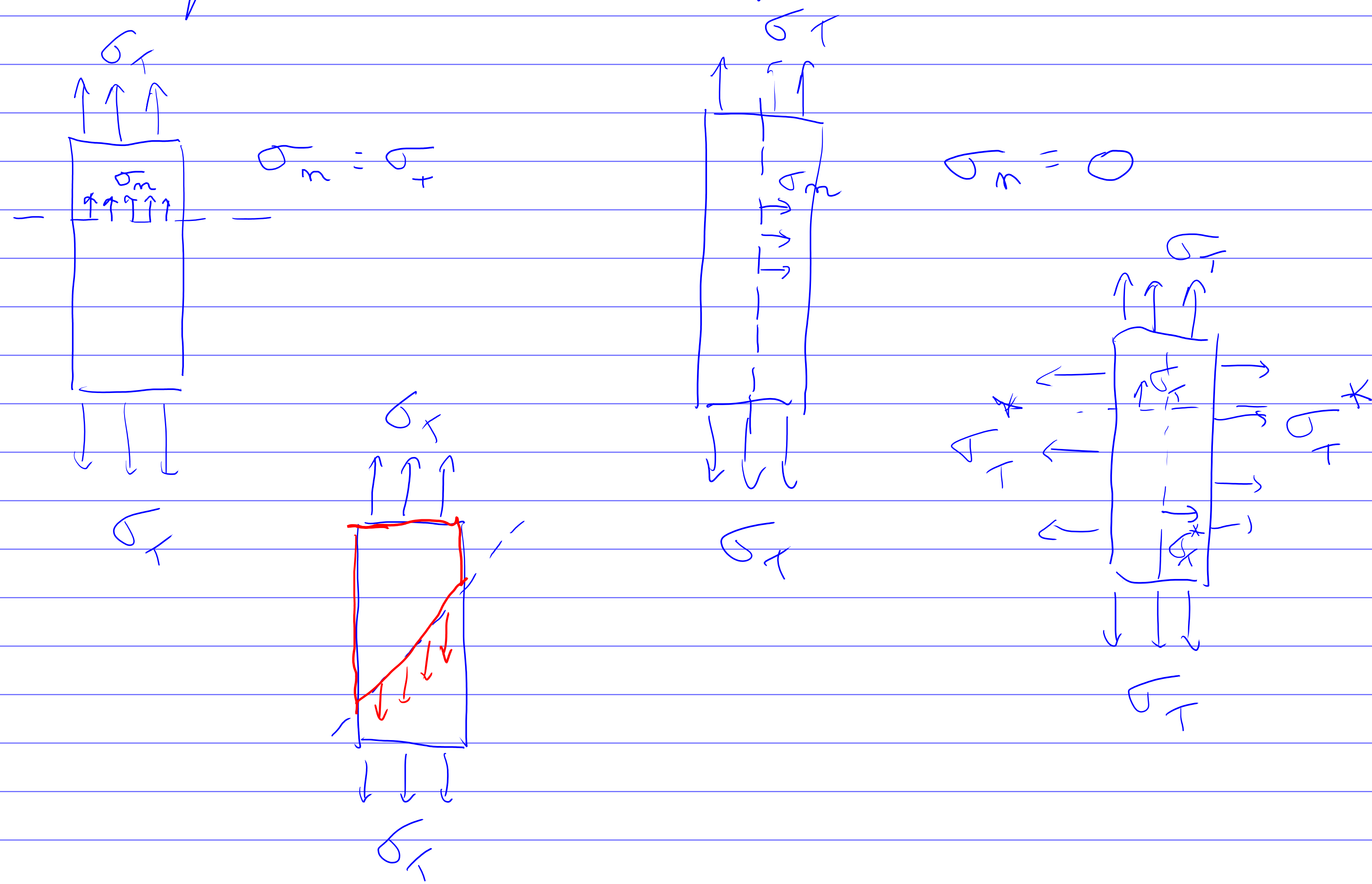
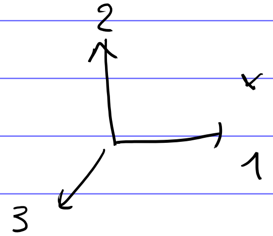
1. Chapitre Contraintes

1) Existence du tenseur des contraintes
2) Equation d'équilibre
3) Symmétrie
4) Commentaires / exemples

$\vec{T}$ dépends de $\vec{n}$ (exemple)



1) Existence

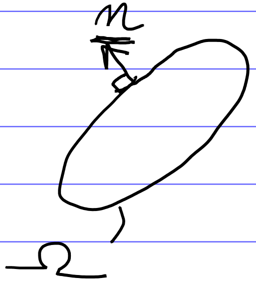


$\times \ \underline{X}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

* Préambule

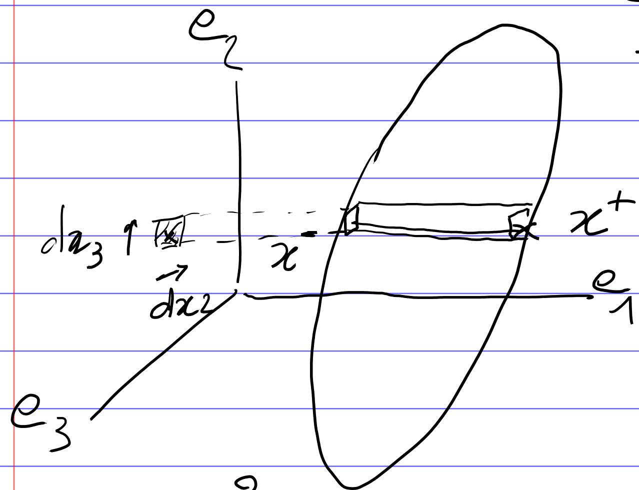
Supposons $f : x_1 \longmapsto f(x_1)$

alors

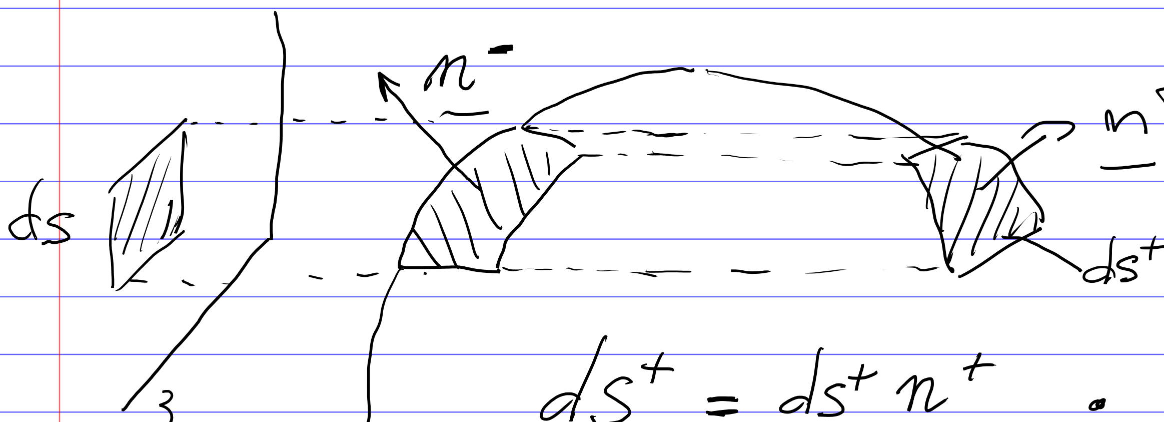


$$\iiint_{\Omega} \frac{df}{dx_1}\, dv = \iint_{\partial\Omega} f\, n_1\, ds$$

Démo :



$$I = \iiint_V \frac{df}{dx_1}\, dv = \iint_{x_2 x_3} \underbrace{\left(\int_{x_1} \frac{df}{dx_1}\, dx_1 \right)}_{= f(x_1^+) - f(x_1^-)} dx_2\, dx_3$$

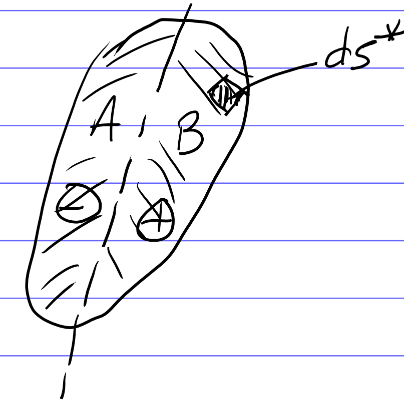


Rmq : Surface projetée de $\underline{ds}$ dans la direction $\underline{n}$ est égale à $\underline{ds} \cdot \underline{n}$

$$\underline{ds}^+ = ds^+\, \underline{n}^+ \quad ; \quad \underline{ds}^- = ds^-\, \underline{n}^-$$

$$ds = \underline{ds}^+ \cdot \underline{e}_1 = \underline{ds}^- \cdot \underline{e}_1$$

$$I = \iint_{x_2 x_3} f(x^+)\,\underline{n}^+ \cdot \underline{e}_1\, ds^+ - \iint_{x_2 x_3} f(x^-)\,\underline{n}^- \cdot \underline{e}_1\, ds^-$$



$$= \iint_{x_2 x_3} f(x)\,\underline{n} \cdot \underline{e}_1\, ds^*$$

Corollaire 1 : si $f = f(x_1, x_2, x_3)$

$$\int_{\Omega} \left(\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3} \right) dv = \int_{\partial\Omega} \left(f\,\underline{n}\cdot\underline{e}_1 + f\,\underline{n}\cdot\underline{e}_2 + f\,n\cdot e_3 \right) ds$$

Corollaire 2 :

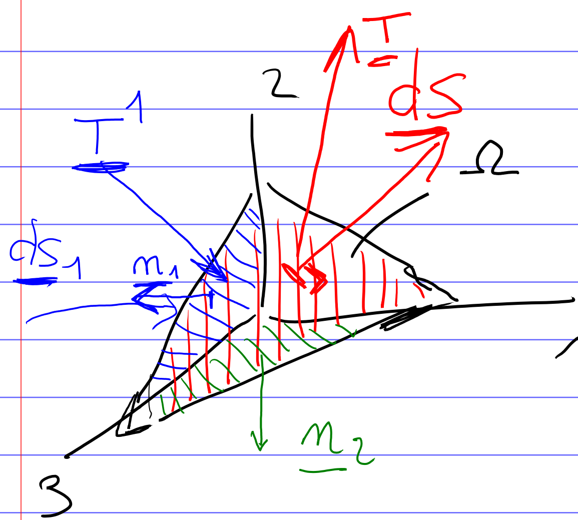
Si $\underline{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ avec $f_k = f_k(x_1, x_2, x_3)$

$$\int_{\Omega} \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) dv = \int_{\partial\Omega} \left(f_1 \underbrace{n_1}_{\underline{n} \cdot \underline{e}_1} + f_2 n_2 + f_3 n_3 \right) ds$$

$$= \int_{\partial\Omega} \underline{f} \cdot \underline{n} \, dS$$

Classiquement "théorème de la divergence" : $\int_{\Omega} \underbrace{\nabla \cdot \underline{f}}_{\text{div.}(\underline{f})} \, dv = \int_{\partial\Omega} \underline{f} \cdot \underline{n} \, ds$

Retour : existence



$\rho \underline{g}$

A l'équilibre : $\sum \vec{F} = 0$

$$ds_1 \times \underline{T}^1 + ds_2 \underline{T}^2 + ds_3 \underline{T}^3 + ds \underline{T}(\underline{n})$$
$$+ \frac{dx_1 dx_2 dx_3}{3} \rho \underline{g} = 0$$

3 premiers termes : forces de surface $O(dx^2)$

dernier terme : force de volume $O(dx^3)$

$\underline{n}_1 = -\underline{e}_1$

$\underline{n}_2 = -\underline{e}_2$

$\underline{n}_3 = -\underline{e}_3$

Dernier terme négligeable car $O(dx^3)$

$$\Rightarrow ds\left(-\underline{n} \cdot \underline{e}_1 \underline{T}^1 + -\underline{n} \cdot \underline{e}_2 \underline{T}^2 + -\underline{n} \cdot \underline{e}_3 \underline{T}^3 + \cancel{\underline{T}}\right) = 0$$

$ds_1 = \underline{ds} \cdot \underline{e}_1$

$$\Rightarrow ds\Big(\underbrace{-\underline{n} \cdot \underline{e}_1}_{ds_1} \underline{T}^1 + -\underline{n} \cdot \underline{e}_2 \underline{T}^2 + \underline{n} \cdot -\underline{e}_3 \underline{T}^3 + \underline{T}\Big) = 0$$

$$\Rightarrow \underline{T} = n_1 \underline{T}^1 + n_2 \underline{T}^2 + n_3 \underline{T}^3$$

$$\underline{T} = \left[\underline{T}^1 \,\vdots\, \underline{T}^2 \,\vdots\, \underline{T}^3\right]_{(e_1, e_2, e_3)} \times \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}_{(e_1, e_2, e_3)}$$

On vient de démontrer qu'il existe une application linéaire tq.

$$\boxed{\underline{T}(\underline{x}, \underline{n}) = \underline{\underline{\sigma}}(\underline{x}) \cdot \underline{n}}$$

On appelle $\underline{\underline{\sigma}}$ le tenseur des contraintes.

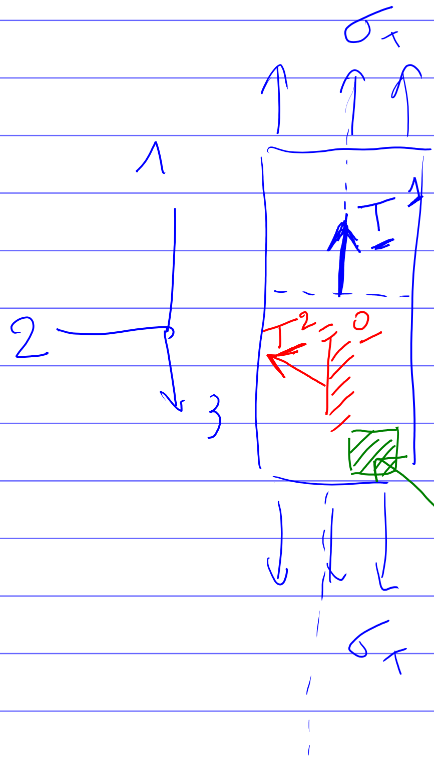
Notation : scalaire : α (tenseur d'ordre 0)

vecteur : $\underline{\alpha}$ (ordre 1)

matrice : $\underline{\underline{\alpha}}$ (ordre 2)

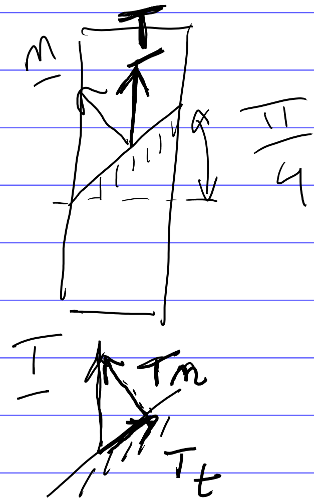
(tenseur d'ordre 3 : $\underline{\underline{\underline{\alpha}}}$ . . .) ← aucune importance ici

* **Exemple**



$\underline{T}^3 = 0$

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

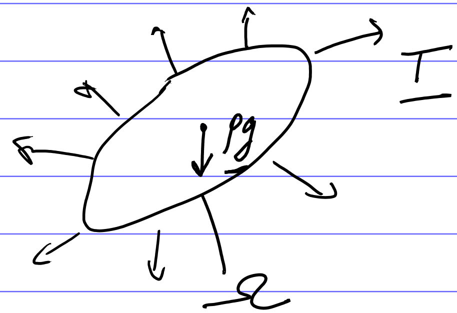


$$\underline{n} : \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\underline{T} = \begin{pmatrix} \frac{1}{\sqrt{2}} \sigma_T \\ 0 \\ 0 \end{pmatrix}$$

* 2) Equilibre

1ère démonstration :



$$\int_{\partial\Omega} \underline{T}\, ds + \int_{\Omega} \rho \underline{g}\, dv = 0$$

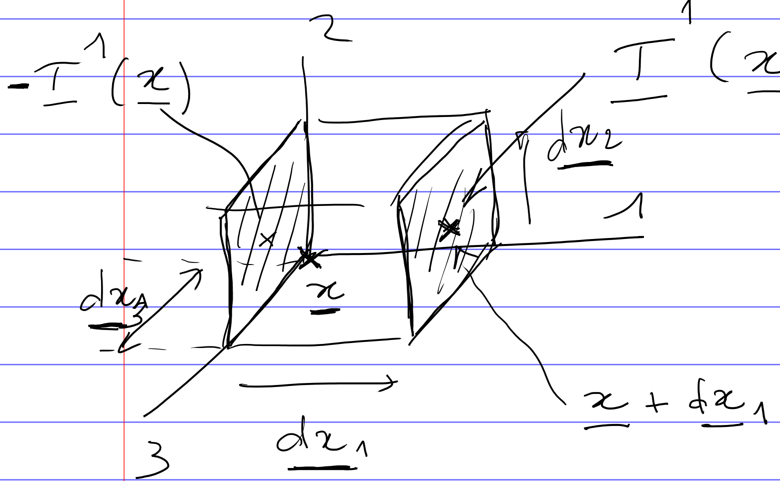
Th. de la divergence : $\displaystyle\int_{\partial\Omega} \underline{T}\, ds = \int_{\partial\Omega} \underline{\underline{\sigma}}\, \underline{n}\, ds = \int_{\Omega} \nabla \cdot \underline{\underline{\sigma}}\, dv$

d'où $\displaystyle\int_{\Omega} (\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{g})\, dv = 0 \quad \forall \Omega$

donc $\boxed{\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{g} = 0}$ Equa. d'équilibre

ou $\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{g} = \rho \ddot{\underline{x}}$ en dynamique

2ème démonstration :



$$\Sigma F = \left(\underline{T}^1(\underline{x}+\underline{dx}_1) - \underline{T}^1(\underline{x})\right) \times dx_2\, dx_3$$

$$+ \left(\underline{T}^2(\underline{x}+\underline{dx}_2) - \underline{T}^2(\underline{x})\right) \times dx_1\, dx_3$$

$$+ \left(\underline{T}^3(\underline{x}+\underline{dx}_3) - \underline{T}^3(\underline{x})\right) \times dx_1\, dx_2$$

$$+ dx_1\, dx_2 + dx_3\, \rho \underline{g} \underset{\text{équilibre}}{=} 0$$

$$\frac{\Sigma F}{V} = \frac{\partial \overline{T}^1}{\partial x_1} \cancel{dx_1} + \frac{\partial \overline{T}^2}{\partial x_2} \cancel{dx_2} + \frac{\partial \underline{T}^3}{\partial x_3}$$

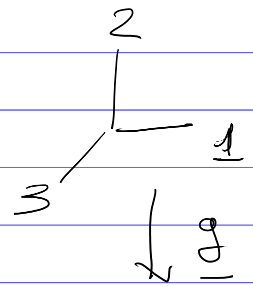
$$+ \rho \underline{g}$$

$$V = dx_1\, dx_2\, dx_3$$

$$\Rightarrow \nabla \cdot \underline{\underline{\sigma}} + \rho \underline{g} = \underline{0}$$

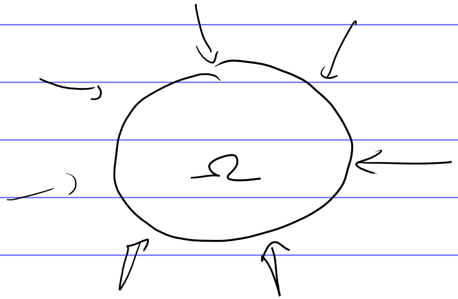
$$\nabla \cdot \underline{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

$$\nabla \cdot \underline{\underline{\sigma}} = \begin{pmatrix} \partial \sigma_{11}/\partial x_1 + \partial \sigma_{12}/\partial x_2 + \partial \sigma_{13}/\partial x_3 \\ \partial \sigma_{12}/\partial x_1 + \partial \sigma_{22}/\partial x_2 + \partial \sigma_{23}/\partial x \\ \partial \sigma_{13}/\partial x_1 + \ldots\ldots \end{pmatrix}$$



Exemple : fluide au repos $\quad \underline{\underline{\sigma}} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$

$$\nabla \cdot \underline{\underline{\sigma}} = -\begin{pmatrix} \partial p/\partial x_1 \\ \partial p/\partial x_2 \\ \partial p/\partial x_3 \end{pmatrix} = -\rho \underline{g} = -\rho \begin{pmatrix} 0 \\ -g \\ 0 \end{pmatrix} \Rightarrow p = -\rho g x_2 + \alpha$$



$$\int_{\partial\Omega} T\,ds = \int_{\Omega} \nabla \sigma\,dv$$

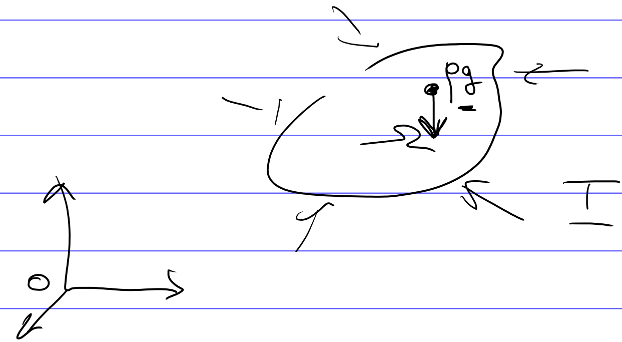
et si $\nabla\cdot\underline{\underline{\sigma}} = -\rho\underline{g}$

$$= \rho\underline{g}\int_{\Omega} dv = \rho_f g\, V_{\Omega}$$

(poussée d'archimède)

* 3) Symmétrie

Sonne des moments



$$\underline{M}_T = \int_{\partial\Omega} \underline{OM} \wedge \underline{T}\, ds \quad , \quad \underline{M}_g = \int_{\Omega} \underline{OM} \wedge \rho \underline{g}\, d\tau$$

$$\underline{M}_T = \int_{\partial\Omega} (\underline{OM} \wedge \underline{\underline{\sigma}}) \cdot \underline{n}\, ds = \int_{\Omega} \nabla \cdot (\underline{OM} \wedge \underline{\underline{\sigma}})\, d\tau$$

$$\left(\underline{OM} \wedge \underline{\underline{\sigma}} = \left(\underline{OM} \wedge \underline{T}^1 \;\middle|\; \underline{OM} \wedge \underline{T}^2 \;\middle|\; \underline{OM} \wedge \underline{T}^3 \right) \right)$$

$$\nabla(\underline{x} \wedge \underline{\underline{\sigma}}) = ?$$

$$\frac{\partial (\underline{OM} \wedge \underline{T}^1)}{\partial x_1} = \begin{pmatrix}1\\0\\0\end{pmatrix} \wedge \underline{T}^1 + \underline{x} \wedge \frac{\partial \underline{T}^1}{\partial x_1}$$

$$\frac{\partial}{\partial x_1}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$\cancel{= \begin{pmatrix} T^2_3 - T^3_2 \\ -T^1_3 + T^3_1 \\ T^1_2 - T^2_1 \end{pmatrix}}$$

$$\nabla\cdot(\underline{x}\wedge\underline{\underline{\sigma}}) = \begin{pmatrix} T^2_3 - T^3_2 \\ T^3_1 - T^1_3 \\ T^1_2 - T^2_1\end{pmatrix} + \underline{x} \wedge \left(\frac{\partial T^1}{\partial x_1} + \frac{\partial T^2}{\partial x_2} + \frac{\partial T^3}{\partial x_3}\right)$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \wedge \underline{T}^1 = \begin{pmatrix}0\\ T^1_3 \\ -T^1_2\end{pmatrix}$$

$$\Rightarrow \underline{M}_t + \underline{M}_g = \int_\Omega \left[\begin{pmatrix}T^2_3 - T^3_2 \\ \dots \\ \dots\end{pmatrix} + x\wedge\left(\underline{\underline{\nabla}}\cdot\sigma + \rho\underline{g}\right)\right] dv$$

$= 0$ à l'équilibre

Vrai pour tout Ω et pour tout choix de l'origine

donc le grand crochet a une intégrale nulle

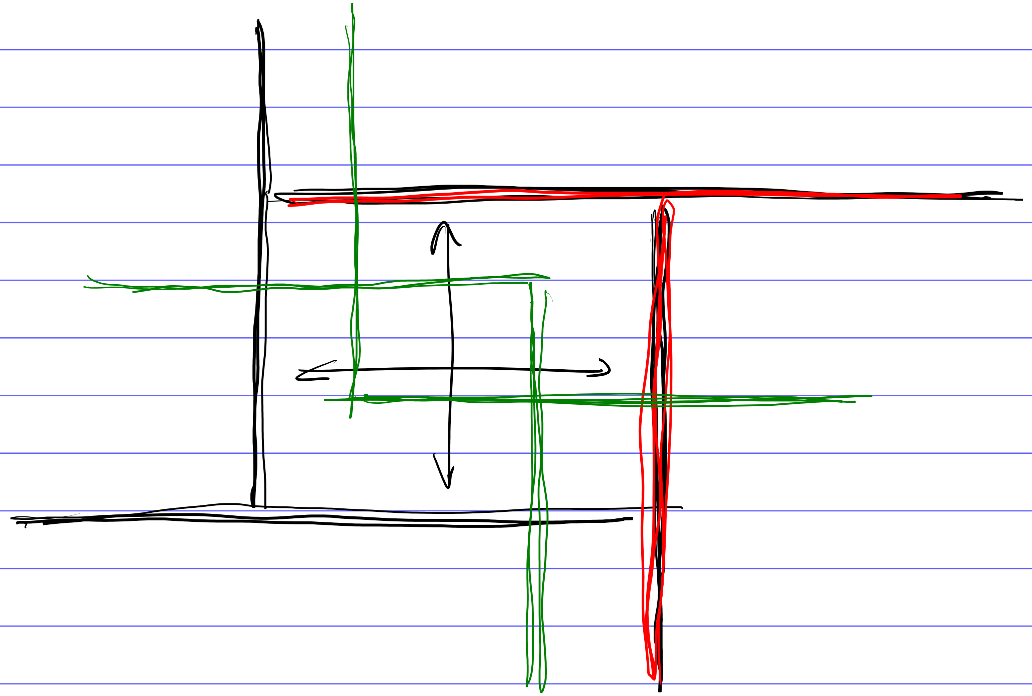
Donc $(\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{g}) = 0$

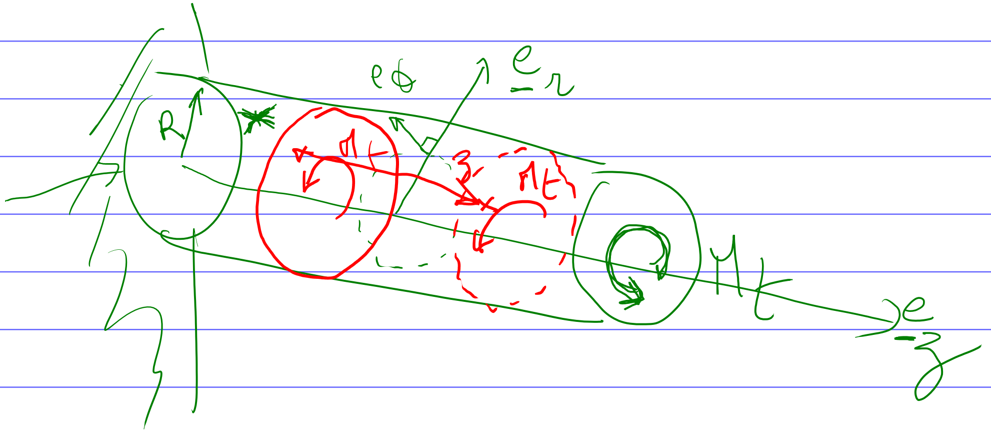
donc $\begin{pmatrix} T_3^{\,2} - T_2^{\,3} \\ \cdots \\ \cdots \end{pmatrix} = \underline{0}$

i.e. $\sigma_{12} = \sigma_{21}$

$\sigma_{13} = \sigma_{31}$

$\sigma_{23} = \sigma_{32}$





1) Ecrire l'équation d'éq.

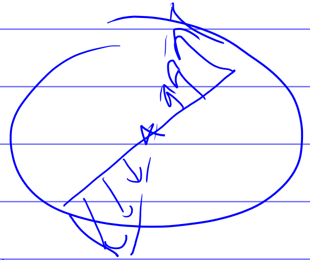
$$\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{g} = \underline{0}$$

2) Conditions aux limites : en tout point de $\partial\Omega$ dire ce qui est imposé

i.e. : $\underline{\underline{\sigma}} \cdot \underline{n} = \vec{T}_{\text{imposé}}$ sur tout $\partial\Omega$

exemple : en $r = R$ $\quad \underline{\underline{\sigma}} \cdot \underline{e}_r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ C.L. "air libre"

3) Rappel de RDM en torsion



$$\boxed{\underline{\tau} = \alpha M_t r \underline{e_\theta}}$$

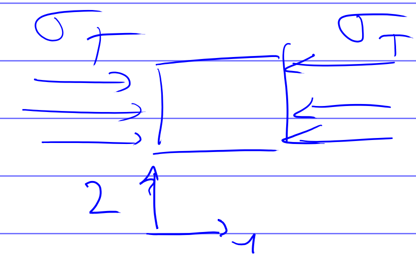
sur une surface $\perp \underline{e_z}$

Comment faire un $\underline{\underline{\sigma}}$ avec ça et est-ce que il satisfait eq. d'équilibre (1) et conditions aux limites (2) ?

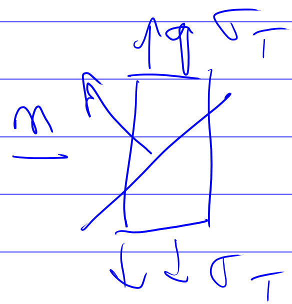
QCM fin de séance

porte sur séance précédente

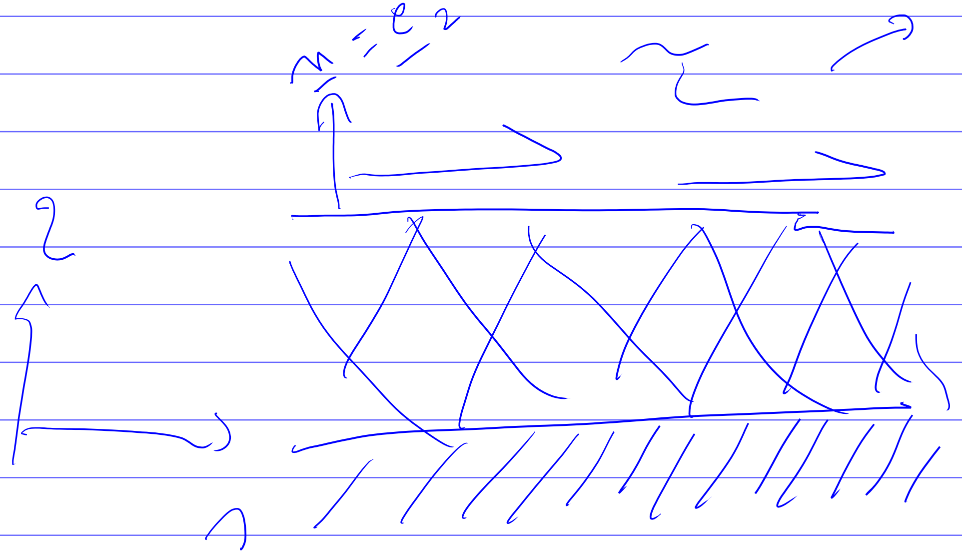
(QCM 1 = contraintes)

Ex de question : 

que vaut σ_{11} ?



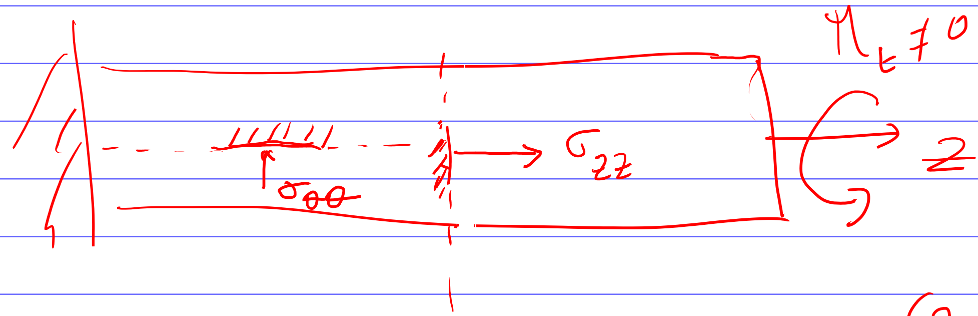
que vaut $\vec{T}(n)$?



$$\vec{T}^2 = \begin{pmatrix}\tau\\0\\0\end{pmatrix}$$

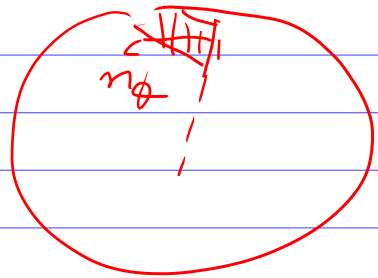
$$\begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{(e_1, e_2, e_3)}$$

Cisaillement pur



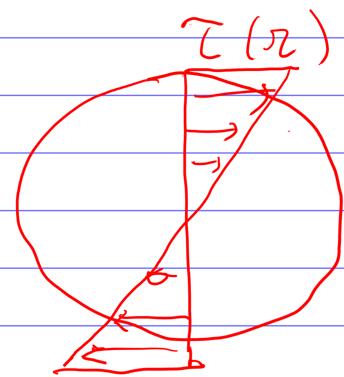
$F = 0 \Rightarrow \sigma_{zz} = 0$

$\sigma_{zz} = 0$ (?)



$T(r, \underline{e}_z) = \tau = \alpha r \underline{e}_\theta$ $\quad \nabla \cdot \sigma$

donne qqch. dans



$\underline{\underline{\sigma}} =$