* Complément cours : ε planes et σ planes.

- Déformations planes : symmétrie par rapport à un plan $(\underline{e}_1, \underline{e}_2)$ et $\varepsilon_{33} = 0$

Exemple : cylindre creux, par hypothèse $u_z = 0$

$\Rightarrow \varepsilon_{zz} = 0$

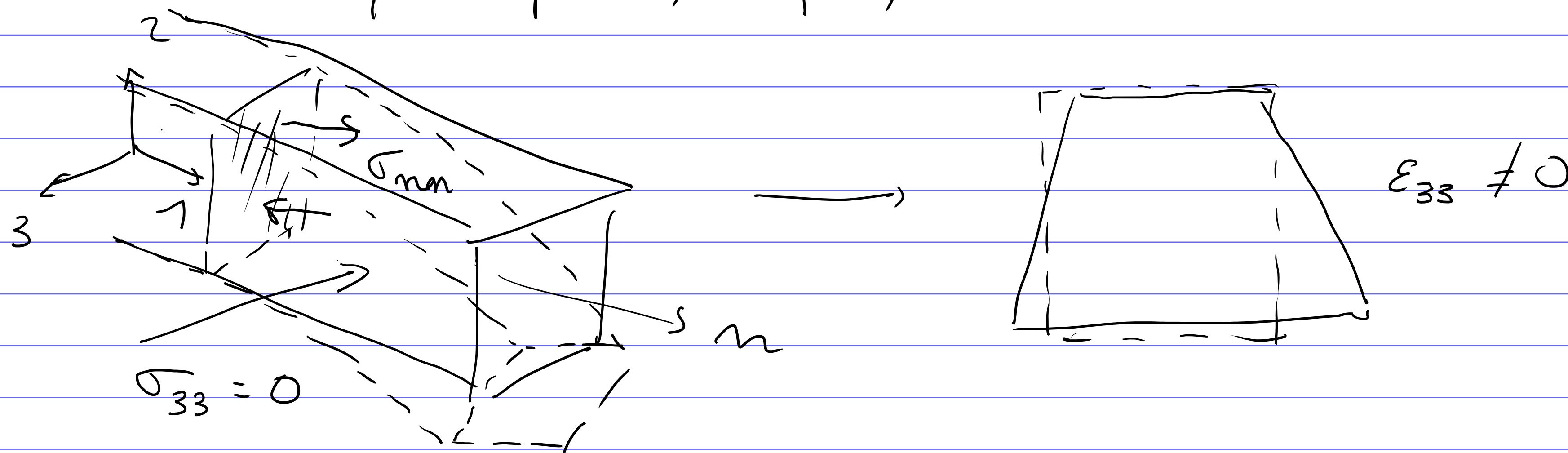
La loi de Hooke implique $\varepsilon_{zz} = \frac{1+\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) - \frac{\nu}{E}\sigma_{33}$

$$\Rightarrow \boxed{\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})}$$

(en déf. planes)

• Contraintes planes : symétrie (e_1, e_2) et $\sigma_{33} = 0$

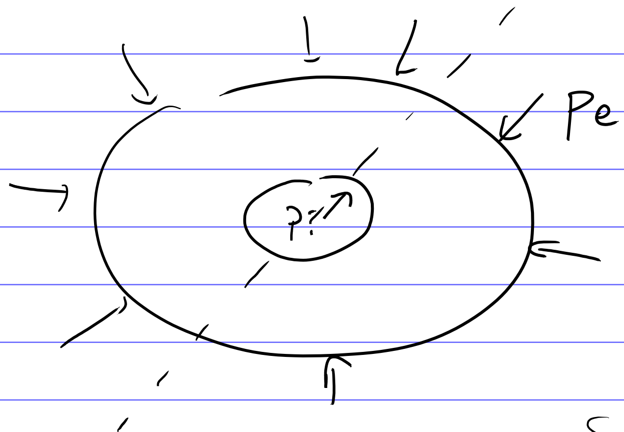
Exemples : poutres, coques, ...



$$\boxed{\varepsilon_{33} = -\frac{\nu}{E}\left(\sigma_{11} + \sigma_{22}\right)}$$

En contraintes planes

Exercice cylindre creux



* Invariance / θ et symmétrie vs. diamètre
* Invariance / z

Seul déplacement : $u_r = u_r(r)$

∇ (grad)
$\nabla\cdot$ (div)
$\nabla\times$ (rot)
$\nabla\wedge$ (fr.)

Equations de Navier :

$$\boxed{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\underbrace{\nabla(\nabla\cdot\underline{u})}_{\text{grad}(\text{div } u)} - \frac{E}{2(1+\nu)}\underbrace{\nabla\times(\nabla\times\underline{u})}_{\text{rot}(\text{rot}(u))} = -\rho\underline{g}}$$

(Forme équivalente à $\nabla\cdot\left(\lambda\,\mathrm{tr}(\underline{\underline{\varepsilon}})\,\underline{\underline{Id}} + 2\mu\underline{\underline{\varepsilon}}\right) = -\rho g$)

$$\sigma = \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix} \ln \varepsilon + 2\mu \begin{bmatrix} \varepsilon_{11} & & \\ & \varepsilon_{22} & \\ & & \varepsilon_{33} \end{bmatrix}$$

$$0 = \sigma_{33} = \lambda \ln \varepsilon + 2\mu \varepsilon_{33}$$

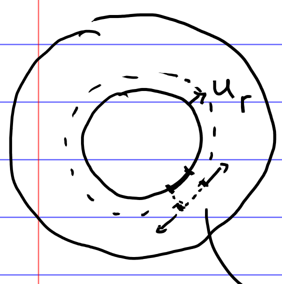
Intérêt de Navier si $\mathrm{div}(\underline{u}) = 0$ ou $\underline{u}$ irrotationel

($\underline{\underline{\nabla}}\,\underline{u}$ est symmétrique) un des termes s'annule.

Ici : $\nabla(\nabla\cdot u) = 0$

se réduit à $\dfrac{d}{dr}\left(\dfrac{1}{r}\dfrac{d}{dr}(ru)\right) = 0$

$$\Rightarrow \quad u = ar + \frac{b}{r}$$



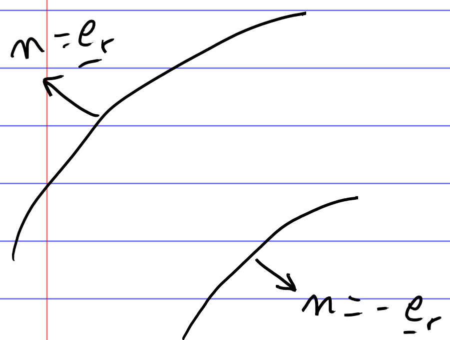
$$\Rightarrow \quad \varepsilon = \begin{bmatrix} \frac{du_r}{dr} & 0 & 0 \\ 0 & \frac{u_r}{r} & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \quad \begin{aligned} \varepsilon_{rr} &= a - \frac{b}{r^2} \\ \varepsilon_{\theta\theta} &= a + \frac{b}{r^2} \end{aligned}$$

$$\varepsilon_{\theta\theta} = \frac{d\mathcal{P}}{\mathcal{P}} = \frac{2\pi\, dr}{2\pi r} = \frac{dr}{r} = \frac{u_r}{r}$$

$\mathcal{P}$: périmètre

$\hookrightarrow \mathrm{tr}(\varepsilon) = 2a$

Hooke : $\begin{cases} \sigma_{rr} = \dfrac{E}{1+\nu}\left(\dfrac{1}{1-2\nu}\,a - \dfrac{b}{r^2}\right) \longleftarrow \text{Cond. lim.} \\[2ex] \sigma_{\theta\theta} = \dfrac{E}{1+\nu}\left(\dfrac{1}{1-2\nu}\,a + \dfrac{b}{r^2}\right) \end{cases}$



$$\sigma_{zz} = \frac{2E\nu}{(1+\nu)(1-2\nu)}\,a$$

Conditions aux lim : $\begin{cases} \underline{\underline{\sigma}}(r=R_e)\cdot\underline{e}_r = -p_e\,\underline{e}_r \\[1ex] \underline{\underline{\sigma}}(r=R_i)\cdot(-\underline{e}_r) = +p_i\,\underline{e}_r \end{cases}$

$$\begin{cases} a = -\dfrac{(1-2\nu)(1+\nu)}{E}\left(\dfrac{R_e^2}{R_e^2-R_i^2}\,p_e - \dfrac{R_i^2}{R_e^2-R_i^2}\,p_i\right) \\[3ex] b = \dfrac{1+\nu}{E}\,\dfrac{R_e^2R_i^2}{R_e^2-R_i^2}\,(p_e-p_i) \end{cases}$$

$$
\begin{cases}
\sigma_{rr} = -\dfrac{R_e^2}{R_e^2 - R_i^2} p_e \left(1 - \dfrac{R_i^2}{r^2}\right) + \dfrac{R_i^2}{R_e^2 - R_i^2} p_i \left(1 - \dfrac{R_e^2}{r^2}\right) \\
\sigma_{\theta\theta} = -\dfrac{R_e^2}{R_e^2 - R_i^2} p_e \left(1 + \dfrac{R_i^2}{r^2}\right) + \dfrac{R_i^2}{R_e^2 - R_i^2} p_i \left(1 + \dfrac{R_e^2}{r^2}\right) \\
\sigma_{zz} = 2\nu \left(\dfrac{R_i^2}{R_e^2 - R_i^2} p_i - \dfrac{R_e^2}{R_e^2 - R_i^2} p_e\right)
\end{cases}
$$

Résultat : dans ce problème en déformation plane

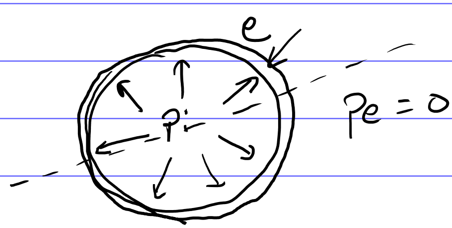
$$
\sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right) = \text{constante}
$$

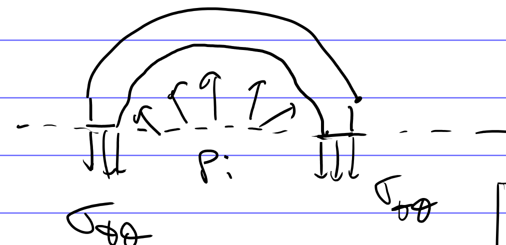
Variation de volume :

$$\frac{\Delta V}{V} = \int_\Omega \varepsilon_v \, dv = \int_\Omega \nabla \cdot \underline{u} \, dv$$

Ici $tr(\varepsilon) = 2a \Rightarrow \boxed{\frac{\Delta V}{V} = 2a}$

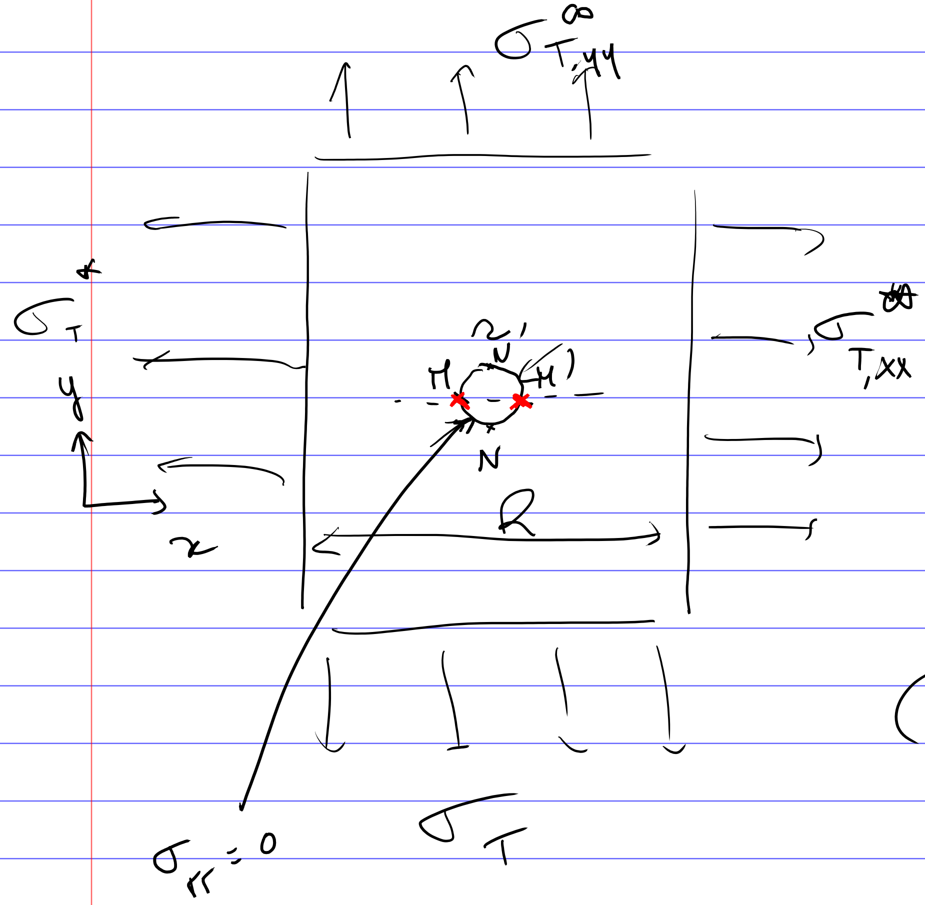
Remarque : Si $R_e - R_i \ll R_e$





$$\langle \sigma_{\phi\phi} \rangle \times 2e - 2p_i R_i = 0$$

$$\boxed{\langle \sigma_{\phi\phi} \rangle \simeq p_i \frac{R_i}{e}}$$



Si $r \ll R \iff$ cylindre creux

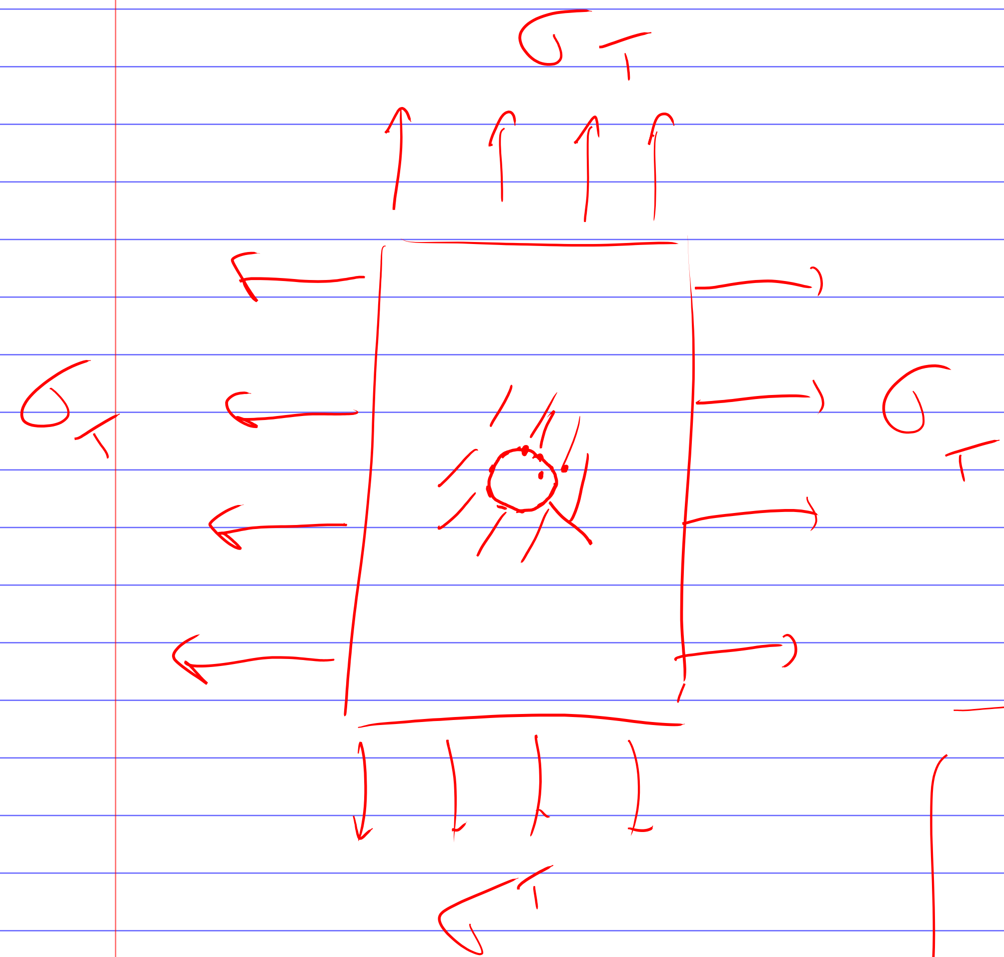
avec $\dfrac{R_e}{R_i} \to +\infty$

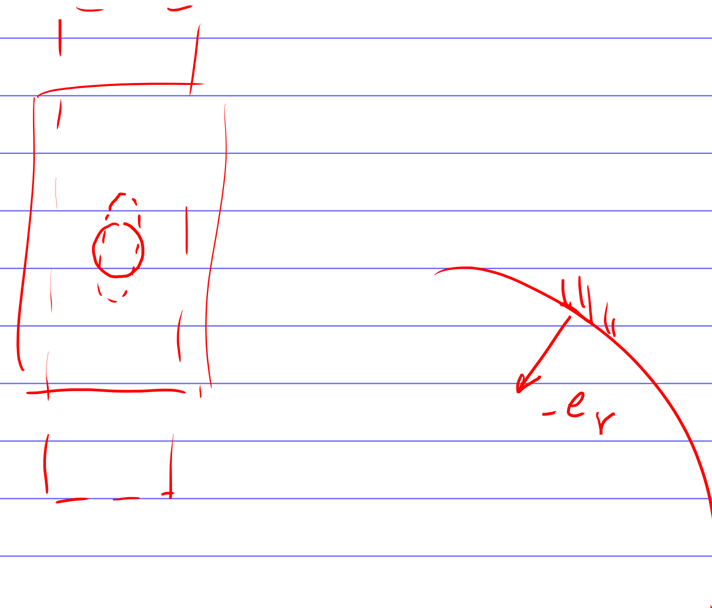
Dans ce cas on obtient si

$p_i = 0$, $\boxed{\sigma_{yy}(M) = 2\,\sigma_{yy}^{\infty}}$

(doublement de la contrainte orthoradiale)

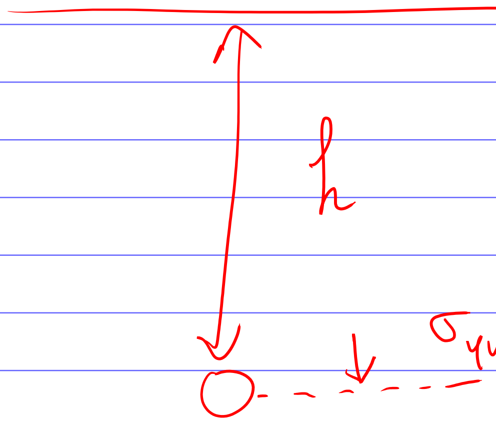
$\boxed{\sigma_{xx}(N) = 2\sigma_{xx}^{\infty}}$





	Avant trou	Après trou
	$\sigma_{rr} = \sigma_T$	$\sigma_{rr} = 0$
	$\sigma_{\theta\theta} = \sigma_T$	$\sigma_{\theta\theta} = 2\sigma_T$

Tunnel



$\sigma_{yy} = h \times \rho g$

A la paroi du tunnel on doit avoir une résistance d'au moins

$\rightarrow \sigma_{yy} = 2h\rho g$

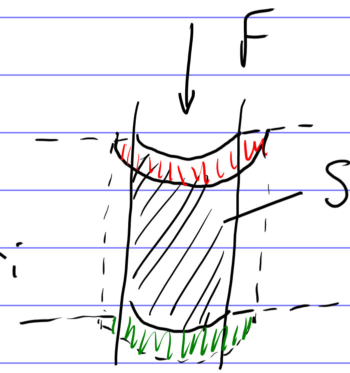
4.2 . Bague cisaillée.

* Invariances et symétries : $\underline{u} = \underline{u}(r) = \begin{pmatrix} u_r(r) \\ 0 \\ u_z(r) \end{pmatrix}$

* Conditions aux limites :

$$\begin{cases} \underline{u} = 0 \text{ en } r = R_e \\ \sigma \cdot (-\underline{e}_r) = \dfrac{F}{S} \underline{e}_z \text{ en } r = R_i \\ \sigma \cdot (\pm \underline{e}_z) = \underline{0} \text{ en haut/bas.} \end{cases}$$

(On pourrait aussi imposer $\underline{u} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}$ en R_i avec "a" une constante, on en déduirait $F = F(a)$)



* Cherchons s'il existe une solution avec $u_r = 0$ partout (s'il y en a une c'est forcément la seule, sinon on tombera sur une contradiction)

$$\underline{u} = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}$$

$$\underline{\nabla} \times \underline{u} = -\frac{du}{dr}\,\underline{u}_\theta$$

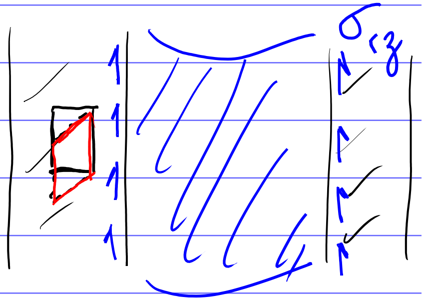
$$\underline{\nabla} \times (\underline{\nabla} \times \underline{u}) = -\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)\underline{u}_z = \underline{0}$$

$$r\frac{du}{dr} = a \quad , \quad u = a \ln\left(\frac{r}{b}\right)$$

CL :
$$\begin{cases} u(r = R_0) = 0 & (1) \\ \overline{T} \text{ imposé en } r_i & (2) \leftarrow \end{cases}$$

Pour (2) :

$$\nabla u = \begin{pmatrix} 0 & 0 & \partial u_z/\partial r \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \varepsilon = \begin{pmatrix} 0 & 0 & \frac{\partial u_z/\partial r}{2} \\ 0 & 0 & 0 \\ \frac{\partial u_z}{\partial r}/2 & 0 & 0 \end{pmatrix}$$



$tr(\varepsilon) = 0$ donc $\sigma = 2\mu\varepsilon$

$\sigma_{rz} = 2\mu a/r$

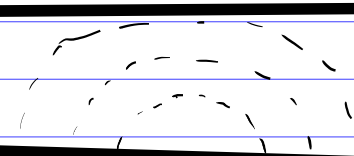
et $F = \sigma_{rz} \times \ell \times 2\pi R_i$

soit (2) $2\mu a / R_i = \dfrac{F}{2\pi \ell R_i} \Rightarrow a = \dfrac{F}{4\mu\pi\ell R_i}$

$0 \quad 0 \quad 1/r$

F

$0 \quad 0 \quad 0$

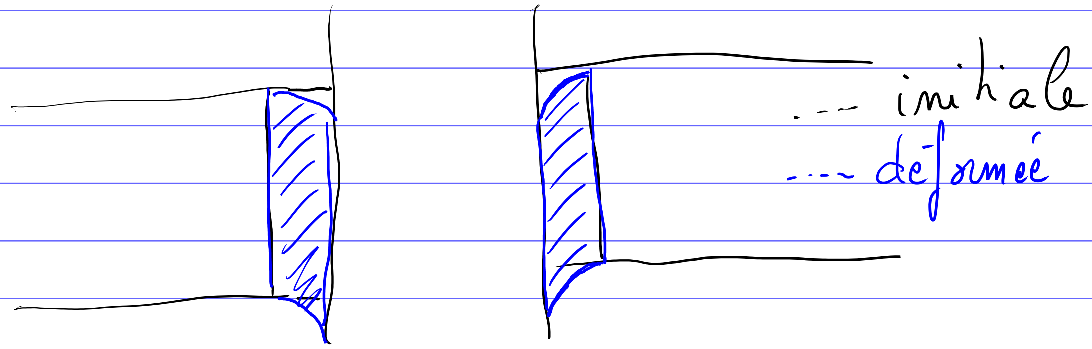


[redacted]

ε

$=$

$=$

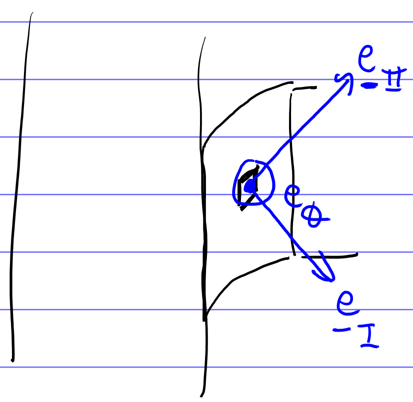


3) Contraintes principales.

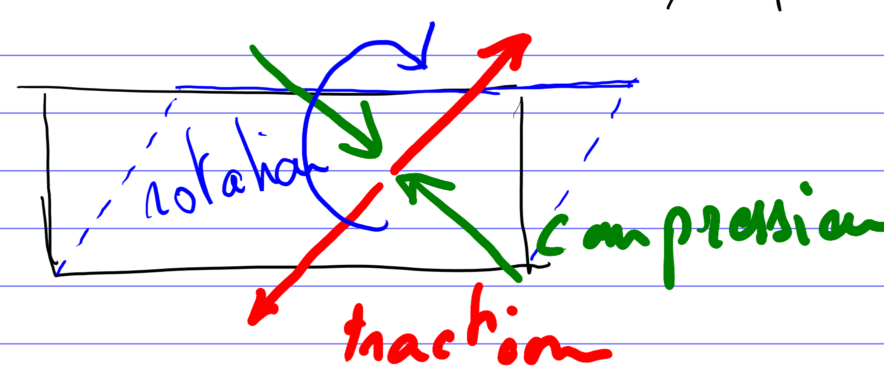
$$\underline{\underline{\sigma}} = \begin{bmatrix} 0 & \sigma_{rz} \\ \sigma_{rz} & 0 \end{bmatrix}_{(r,z)} \qquad \det(\underline{\underline{\sigma}} - \lambda \underline{\underline{I_d}}) = \begin{vmatrix} -\lambda & \sigma_{rz} \\ \sigma_{rz} & -\lambda \end{vmatrix}$$

nul si $\lambda = +/- \, \sigma_{rz}$

vecteurs propres : $\underline{e}_I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\underline{e}_{II} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



Ce qui revient à dire que $\begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix}$ représente une combinaison traction/compression



$$\begin{bmatrix} 0 & 0 & \tau \\ 0 & 0 & 0 \\ \tau & 0 & 0 \end{bmatrix}_{(r,\Theta,z)} \rightarrow \begin{bmatrix} \tau & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tau \end{bmatrix}_{(e_I,\Theta,e_{II})}$$

(cisaillement simple) (traction/compression)

4.3 Auto-gravitation

$$\underline{u} = u(r)\,\underline{e}_r$$

$$\underline{\nabla} \times \underline{u} = \underline{0} \quad \text{(irrotationnel)}$$

$$\Rightarrow \begin{cases} \dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}\,\underline{\nabla}(\nabla \cdot \underline{u}) + \underline{b} = 0 \\ \underline{b} = -\rho g \dfrac{r}{R}\,\underline{e}_r \end{cases}$$

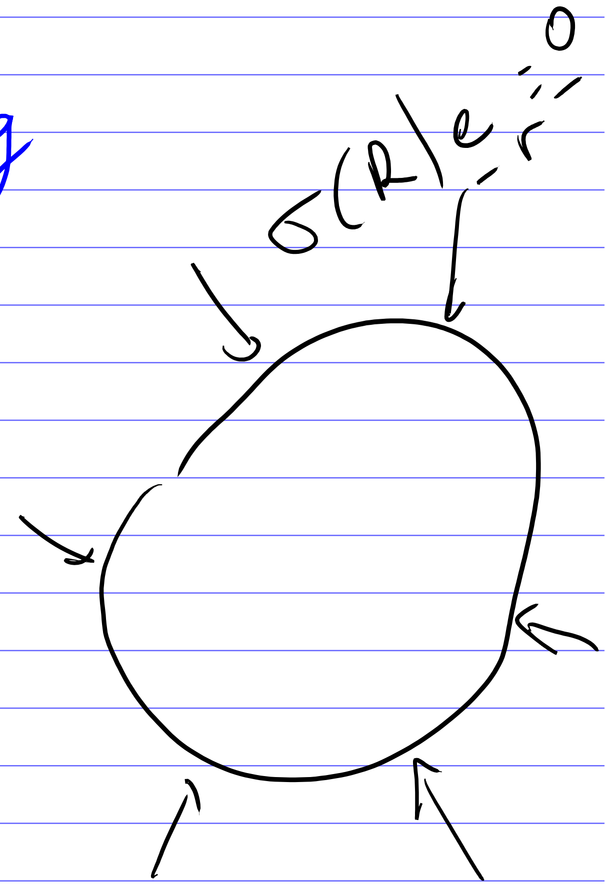
$$\underline{\nabla}(\nabla\cdot\underline{u}) = \frac{d}{dr}\left(\frac{1}{r^2}\frac{d}{dr}\left(r^2 u\right)\right)\underline{e}_r = \alpha r\, \underline{e}_r$$

$$\alpha = \frac{(1+\nu)(1-2\nu)}{(1-\nu)ER}\rho g$$

$$u = \frac{\alpha}{10} r^3 + a r + \frac{b}{r^2}$$

$$u(r=0) = 0 \quad \Rightarrow \quad b = 0$$

$$\underline{\underline{\sigma}}(r=R)\cdot\underline{e}_r = \underline{0} \quad \Rightarrow \quad a = ?$$



$$\varepsilon = \begin{pmatrix} \frac{3\alpha r^2}{10} + a & 0 & 0 \\ 0 & \frac{\alpha r^2}{10} + a & 0 \\ 0 & 0 & \frac{\alpha r^2}{10} + a \end{pmatrix}$$

↳ loi de Hooke $\longrightarrow a = \dfrac{(3-\nu)(1-2\nu)}{10(1-\nu)} \dfrac{\rho g R}{E}$

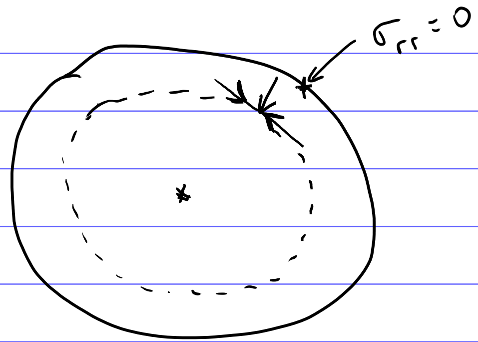
1) * $\varepsilon_{rr} < 0$ pour $r < R\sqrt{\dfrac{3-\nu}{3(1+\nu)}}$ ← deux zones distinctes.

* $\varepsilon_{\theta\theta} < 0$ pour $r < R\sqrt{\dfrac{3-\nu}{1+\nu}}$ ← toujours vrai puisque $\nu < 0{,}5$

$\varepsilon_{\varphi\varphi} = \varepsilon_{\theta\theta}$

2. $$\frac{\Delta V}{V} = \frac{\alpha R^2}{10}\left(1 - \frac{3(3-\nu)}{1+\nu}\right) \quad (= 0 \text{ si } \nu = 0{,}5)$$

3. au centre $$p = \frac{3-\nu}{10(1-\nu)}\rho g R$$



suivant θ : compression, suivant r : $\sigma_{rr} = 0$
(à la surface)

Hooke $\Rightarrow \varepsilon_{rr} = -\frac{\nu}{E}(\sigma_{\theta\theta} + \sigma_{\varphi\varphi})$